13 Vectors, Lines and Planes

Stefan Banach

Stefan Banach was born in Kraków, Poland (at the time part of the Austro-Hungarian Empire) on 30 March 1892. During his early life Banach was brought up by Franciszka Plowa, who lived in Kraków with her daughter Maria. Maria's guardian was a French intellectual Juliusz Mien, who quickly recognised Banach's talents. Mien gave Banach a general education, including teaching him to speak French, and in general gave him an appreciation for education. On leaving school Banach chose to study engineering and went to Lvov, which is now in the Ukraine, where he enrolled in the Faculty of Engineering at Lvov Technical University.

By chance, Banach met another Polish mathematician, Steinhaus, in 1916. Steinhaus told Banach of a mathematical problem that he was working on without making much headway and after a few days Banach had the main idea for the required counter-example, which led to Steinhaus and Banach writing a joint paper. This was Banach's first paper and it was finally published in 1918. From then on he continued to publish and in 1920 he went to work at Lvov Technical University. He initially worked as an assistant lecturing in mathematics and gaining his doctorate, but was promoted to a full professorship in 1924. Banach worked in an unconventional manner and was often found doing mathematics with his colleagues in the cafés of Lvov. Banach was the founder of modern functional analysis, made major contributions to the theory of topological vector spaces and defined axiomatically what today is called a Banach space, which is a real or complex normed vector space.

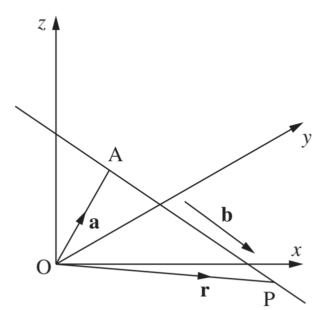
At the beginning of the second world war Soviet troops occupied Lvov, but Banach was allowed to continue at the university, and he became the Dean of the Faculty of Science. However, the Nazi occupation of Lvov in June 1941 ended his career in university and by the end of 1941 Banach was working in the German institute dealing with infectious diseases, feeding lice. This was to be his life during the remainder of the Nazi occupation of Lvov in July 1944. Once the Soviet troops retook Lvov, Banach renewed his contacts, but by this time was seriously ill. Banach had planned to take up the chair of mathematics at the Jagiellonian University in Kraków, but he died of lung cancer in Lvov in 1945.

13.1 Equation of a straight line

Vector equation of a straight line

In both two and three dimensions a line is described as passing through a fixed point and having a specific direction, or as passing through two fixed points.

Vector equation of a line passing through a fixed point parallel to a given vector



Consider the diagram.

The fixed point A on the line has position vector $\overrightarrow{OA} = \mathbf{a}$.

The line is parallel to vector **b**.

Any point P(x, y, z) on the line has position vector $\overrightarrow{OP} = \mathbf{r}$.

Hence $\overrightarrow{AP}$ has direction **b** and a variable magnitude: that is, $\overrightarrow{AP} = \lambda\mathbf{b}$, where λ is a variable constant.

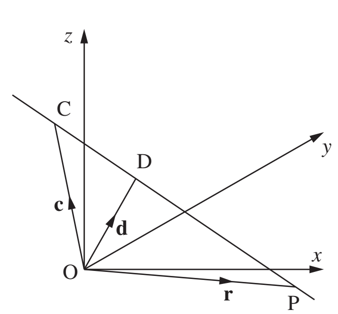
Using vector addition, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ so $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

This is the vector equation of a line, and **b** is known as the **direction vector** of the line.

Alternatively, if $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then we could write the vector equation of a line as: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

If we put in values for λ we then get the position vectors of points that lie on the line.

Vector equation of a line passing through two fixed points



Consider the diagram.

One fixed point on the line is C, which has position vector $\overrightarrow{OC} = \mathbf{c}$, and the second fixed point on the line is D, which has position vector $\overrightarrow{OD} = \mathbf{d}$. Again any point P$(x, y, z)$ on the line has position vector $\overrightarrow{OP} = \mathbf{r}$.

Hence $\overrightarrow{CP}$ has direction $\mathbf{d} - \mathbf{c}$ and variable magnitude: that is, $\overrightarrow{CP} = \lambda(\mathbf{d} - \mathbf{c})$.

Therefore $\mathbf{r} = \mathbf{c} + \lambda(\mathbf{d} - \mathbf{c})$.

This is the vector equation of a line passing though two fixed points.

We rarely use the vector equation in this form. It is usually easier to find the direction vector then use the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

Example

Find the vector equation of the line passing through the point A (1, 3, 4) parallel to the vector $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

The position vector of A is $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

Hence the vector equation of the line is $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$.

It is essential not to forget the “**r**” in the equation.

Example

Find the vector equation of the line passing through A (3, 2) parallel to the vector $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Even though this is in two dimensions it works in exactly the same way, since we are assuming that the z-component of the vector is zero.

The position vector of A is $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Hence the vector equation of the line is $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Example

Find the vector equation of the line passing though $A(2, 3, -1)$ and $B(4, -2, 2)$.

There are a number of ways of tackling this question.

Method 1

If we let A have position vector **a** and let B have position vector **b**, then we can say that $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

Now $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$

Hence $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4-2 \\ -2-3 \\ 2-(-1) \end{pmatrix}$

$\Rightarrow \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$

Method 2

We could use **b** as a position vector instead of **a**, giving $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$.

Method 3

We could begin by forming the direction vector.

The direction vector is $\mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$.

We could also use $\mathbf{a} - \mathbf{b}$ as the direction vector.

Hence the vector equation of the line is $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$

or $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$.

These two equations are equivalent.

Parametric equations of a straight line

The parametric equations of a straight line are when the vector equation is expressed in terms of the parameter λ. This form is often used when we are doing calculations using the vector equation of a line.

λ is not always used as the parameter. Other common letters are s, t, m, n and μ.

The way to find the parametric equations is shown in the following example.

Example

Give the vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(6\mathbf{i} - 7\mathbf{j} + 3\mathbf{k})$ in parametric form.

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(6\mathbf{i} - 7\mathbf{j} + 3\mathbf{k})$ can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(6\mathbf{i} - 7\mathbf{j} + 3\mathbf{k})$$

$$= (1 + 6\lambda)\mathbf{i} + (2 - 7\lambda)\mathbf{j} + (-4 + 3\lambda)\mathbf{k}$$

Equating components gives the parametric equations:

$x = 1 + 6\lambda$
$y = 2 - 7\lambda$
$z = -4 + 3\lambda$

Example

Find the parametric equations of the line passing through the points $A(-2, 1, 3)$ and $B(1, -1, 4)$.

We begin by forming the vector equation.

The direction vector of this equation is $\overrightarrow{BA} = \begin{pmatrix} 1-(-2) \\ -1-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Hence the vector equation is $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 3\mu \\ -1 + 2\mu \\ 4 + \mu \end{pmatrix}$$

Equating components gives the parametric equations:

$x = 1 + 3\mu$
$y = -1 + 2\mu$
$z = 4 + \mu$

Example

Show that the point $(3, 6, -1)$ fits on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ but $(2, 3, 0)$ does not.

To do this it is easiest to use the parametric forms of the equation.

Therefore $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + \lambda \\ 3\lambda \\ -1 \end{pmatrix}$

The parametric equations of this line are:

$x = 1 + \lambda$
$y = 3\lambda$
$z = -1$

Using $x = 1 + \lambda$ we can see that if $x = 3$ then $\lambda = 2$.

We now check this is consistent with the values of y and z.

When $\lambda = 2$, $y = 6$ and $z = -1$.

z is not dependent on λ; it is always -1.

Hence $(3, 6, -1)$ lies on the line.

Now if $x = 2$, $\lambda = 1$.

Since $y = 3\lambda$, $y = 3$.

However, $z = -1$ and hence $(2, 3, 0)$ does not lie on the line.

Example

Find the coordinates of the point where the line passing through A$(2, 3, -4)$ parallel to $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ crosses the xy-plane.

The easiest way to solve a problem like this is to put the equation in parametric form.

The vector equation of the line is $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

Hence $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

$$= (2 - \lambda)\mathbf{i} + (3 - 3\lambda)\mathbf{j} + (-4 + \lambda)\mathbf{k}$$

Therefore the parametric equations of the line are:

$x = 2 - \lambda$
$y = 3 - 3\lambda$
$z = -4 + \lambda$

This line will cross the xy-plane when $z = 0$.

Hence $\lambda = 4$, giving $x = -2$ and $y = -9$.

Thus the coordinates of the point of intersection are $(-2, -9, 0)$.

Cartesian equations of a straight line

Consider these parametric equations of a straight line:

$x = 3\mu + 1$
$y = 2\mu - 1$
$z = 4 + \mu$

If we now isolate μ we find:

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-4}{1} (= \mu)$$

These are known as the **Cartesian equations of a line** and are in fact a three-dimensional version of $y = mx + c$. In a general form this is written as $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

where (x_0, y_0, z_0) are the coordinates of a point on the line and $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is the direction vector of the line.

Example

A line is parallel to the vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and passes through the point $(2, -3, 5)$. Find the vector equation of the line, the parametric equations of the line, and the Cartesian equations of the line.

The vector equation of the line is given by $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

To form the parametric equations we write

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= (2 + 2s)\mathbf{i} + (-3 - s)\mathbf{j} + (5 + 2s)\mathbf{k}$$

Equating the coefficients of x, y and z gives the parametric equations:

$x = 2 + 2s$
$y = -3 - s$
$z = 5 + 2s$

Eliminating s gives the Cartesian equations:

$$\frac{x-2}{2} = \frac{y+3}{-1} = \frac{z-5}{2} (= s)$$

Sometimes we need to undo the process, as shown in the next example.

Example

Convert the Cartesian equations $\frac{2x-5}{2} = \frac{3-y}{3} = \frac{-2z+5}{5}$ to parametric and vector form.

We begin by writing $\frac{2x-5}{2} = \frac{3-y}{3} = \frac{-2z+5}{5} = t$, where t is a parameter.

Hence $x = \frac{2t+5}{2}$

$y = 3 - 3t$

$z = \frac{5t-5}{-2}$

These are the parametric equations of the line.

Now $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2t+5}{2} \\ 3 - 3t \\ \frac{5t-5}{-2} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 3 \\ \frac{5}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ -\frac{5}{2} \end{pmatrix}$$

Separating the parameter t

Hence $\mathbf{r} = \begin{pmatrix} 2.5 \\ 3 \\ 2.5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ -2.5 \end{pmatrix}$ is the vector equation of the line.

Exercise 1

1 Find the vector equation of the line that is parallel to the given vector and passes through the given point.

a Vector $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, point $(0, 2, -3)$

b Vector $\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$, point $(1, -2, 0)$

c Vector $\begin{pmatrix} 0 \\ -5 \\ 12 \end{pmatrix}$, point $(4, 4, 3)$

d Vector $3\mathbf{i} + 6\mathbf{j} - \mathbf{k}$, point $(5, 2, 1)$

e Vector $2\mathbf{i} - \mathbf{j}$, point $(-3, -1)$

f Vector $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$, point $(-5, 1)$

2 Find the vector equation of the line passing through each pair of points.

a $(2, 1, 2)$ and $(-2, 4, 3)$ **b** $(-3, 1, 0)$ and $(4, -1, 2)$

c $(2, -2, 3)$ and $(0, 7, -3)$ **d** $(3, 4, -2)$ and $(2, -5, -1)$

e $(4, -3)$ and $(1, -3)$

3 Write down equations, in vector form, in parametric form and in Cartesian form, for the line passing through point A with position vector **a** and direction vector **b**.

a $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ $\mathbf{b} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

b $\mathbf{a} = \begin{pmatrix} -3 \\ -2 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix}$

c $\mathbf{a} = \mathbf{j} + \mathbf{k}$ $\mathbf{b} = \mathbf{i} - 3\mathbf{k}$

d $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

4 Convert these vector equations to parametric and Cartesian form.

a $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$

b $\mathbf{r} = 2\mathbf{i} - 5\mathbf{j} - \mathbf{k} + \mu(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

c $\mathbf{r} = \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} + m\begin{pmatrix} 4 \\ -7 \\ 6 \end{pmatrix}$

d $\mathbf{r} = \mathbf{i} - \mathbf{j} + 7\mathbf{k} + n(2\mathbf{i} - 3\mathbf{j} - \mathbf{k})$

e $\mathbf{r} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + s\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

f $\mathbf{r} = \mathbf{i} - 6\mathbf{j} + t(2\mathbf{i} - 5\mathbf{j})$

5 Convert these parametric equations to vector form.

a $x = 3\lambda - 7$, $y = \lambda + 6$, $z = 2\lambda + 4$

b $x = -\mu - 4$, $y = 3\mu - 5$, $z = 5\mu + 1$

c $x = 5m + 4$, $y = 3m$, $z = 4m - 3$

d $x = -4$, $y = 2n - 1$, $z = 5 - 2n$

6 Convert these Cartesian equations to vector form.

a $\dfrac{x-3}{4} = \dfrac{y+5}{3} = \dfrac{z+1}{-3}$ **b** $\dfrac{2x-5}{4} = \dfrac{3y+3}{-4} = \dfrac{z+1}{2}$

c $\dfrac{2-5x}{-4} = \dfrac{3y+5}{-6} = \dfrac{2z+7}{3}$ **d** $\dfrac{6x+1}{4} = \dfrac{4-3y}{-2} = \dfrac{4-z}{6}$

e $5 - 3x = \dfrac{2+3y}{-2} = \dfrac{3z-1}{-3}$ **f** $\dfrac{x-5}{7} = \dfrac{3-7y}{4}; z = 2$

7 Determine whether the given point lies on the line.

a $(-11, 4, -11)$ $\mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

b $(7, -16, 15)$ $x = 2\mu - 3, y = 4 - 4\mu, z = 3\mu$

c $(4, 3, 9)$ $x = 5 - m, y = 2m + 1, z = 3 + 5m$

d $(11, 33, 12)$ $\dfrac{x+4}{3} = \dfrac{y+2}{7} = \dfrac{2z+1}{5}$

e $\left(8, 10, \dfrac{7}{3}\right)$ $\dfrac{2x-4}{3} = \dfrac{3y-5}{7} = \dfrac{3z+1}{2}$

f $\left(-\dfrac{1}{2}, -\dfrac{5}{2}, -3\right)$ $\dfrac{2x+5}{2} = \dfrac{3-2y}{4}; z = -3$

8 The Cartesian equation of a line is given by $\dfrac{3x-5}{6} = \dfrac{2-y}{3} = \dfrac{3z+1}{2}$.

Find the vector equation of the parallel line passing through the point with coordinates $(3, 7, -1)$ and find the position vector of the point on this line where $z = 0$.

9 Find the coordinates of the points where the line $\mathbf{r} = \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ intersects the xy-, the yz- and the xz-planes.

10 Write down a vector equation of the line passing through A and B if

a $\overrightarrow{OA}$ is $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\overrightarrow{OB}$ is $8\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$

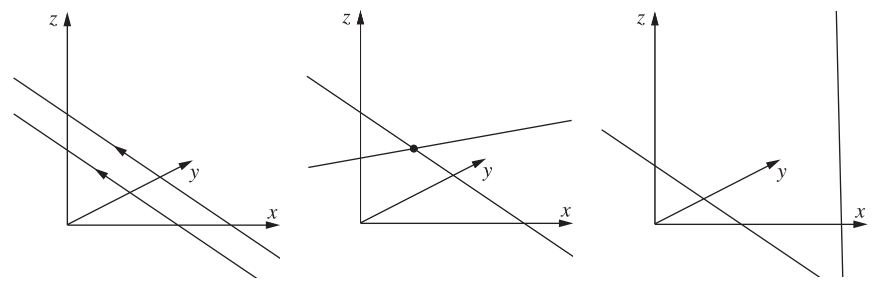
b A and B have coordinates $(2, 6, 7)$ and $(4, 4, 5)$.

In each case, find the coordinates of the points where the line crosses the xy-plane, the yz-plane and the xz-plane.

11 Find where the line $\dfrac{3x+9}{4} = \dfrac{2-y}{-1} = \dfrac{4-3z}{5}$ intersects the xy-, the yz- and the xz-planes.

13.2 Parallel, intersecting and skew lines

When we have two lines there are three possible scenarios. These are shown below.



Parallel lines   Intersecting lines   Skew lines

Skew lines can only exist in three (or more) dimensions.

It is important to consider these three different cases.

Parallel lines

This is the simplest case, and here the direction vector of one line will be the same as or a multiple of the other. It is only the direction vectors of the line that we need to consider here.

Example

State whether the lines $\mathbf{r} = 2\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + s(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k} + t(4\mathbf{i} - 12\mathbf{j} - 8\mathbf{k})$ are parallel or not, giving a reason.

Since $4\mathbf{i} - 12\mathbf{j} - 8\mathbf{k} = -4(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ the lines are clearly parallel.

Example

Show that the following lines are parallel.

$$\text{Line 1}\begin{cases} x = 3s - 1 \\ y = 8s + 3 \\ z = 6s + 1 \end{cases} \quad \text{Line 2}\begin{cases} x = 1.5t + 4 \\ y = 4t - 7 \\ z = 3t - 4 \end{cases}$$

In parametric form, all we need to consider are the coefficients of the parameter as these are the direction vectors of the lines.

Hence the direction vector of Line 1 is $\begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix}$ and the direction vector of Line 2 is $\begin{pmatrix} 1.5 \\ 4 \\ 3 \end{pmatrix}$.

Since $\begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1.5 \\ 4 \\ 3 \end{pmatrix}$ the lines are parallel.

Example

Show that the lines $\frac{x-3}{2} = \frac{1-y}{4} = \frac{2z+1}{3}$ and $\frac{2x+1}{4} = \frac{y+3}{-4} = \frac{4z-5}{6}$ are parallel.

It is tempting to just look at the denominators and see if they are multiples of one another, but in this case it will give the wrong result.

For the denominators to be the direction vectors of the line they must be in a form with positive unitary coefficients of x, y and z.

Hence Line 1 is $\frac{x-3}{2} = \frac{y-1}{-4} = \frac{z+\frac{1}{2}}{\frac{3}{2}}$ and Line 2 is $\frac{x+\frac{1}{2}}{2} = \frac{y+3}{-4} = \frac{z-\frac{5}{4}}{\frac{6}{4}}$.

By comparing the denominators, which are now the direction vectors of the line, we see they are the same, as $\frac{6}{4} = \frac{3}{2}$, and hence the lines are parallel.

This method can also be used to find whether two lines are coincident, that is, they are actually one line.

To show two lines are coincident

To do this we need to show that the lines are parallel and that they have a point in common.

Example

Show that the lines $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -11 \end{pmatrix} + t\begin{pmatrix} 5 \\ 5 \\ -20 \end{pmatrix}$ are coincident.

Since $-5\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ -20 \end{pmatrix}$ the lines are parallel.

We know the point $(1, 2, -3)$ lies on the first line. We now test whether it lies on the second line.

The parametric equations for the second line are:

$x = 3 + 5t$
$y = 4 + 5t$
$z = -11 - 20t$

Letting $3 + 5t = 1$

$$\Rightarrow t = -\frac{2}{5}$$

Substituting $t = -\frac{2}{5}$ into the equations for y and z gives $y = 4 + 5\left(-\frac{2}{5}\right) = 2$ and $z = -11 - 20\left(-\frac{2}{5}\right) = -3$.

Hence the point $(1, 2, -3)$ lies on both lines and the lines are coincident.

Intersecting and skew lines

These two cases are treated as a pair. In neither case can the lines be parallel, so we first need to check that the direction vectors are not the same or multiples of each other. Provided the lines are not parallel, then they either have a common point, in which case they intersect, or they do not, in which case they are skew.

Method

1. Check the vectors are not parallel and put each line in parametric form. Make sure the parameters for each line are different.
2. Assume that they intersect, and equate the *x*-values, the *y*-values and the *z*-values.
3. Solve a pair of equations to calculate the values of the parameter.
4. Now substitute into the third equation. If the parameters fit, the lines intersect, and if they do not, the lines are skew. To find the coordinates of intersection, substitute either of the calculated parameters into the parametric equations.

Example

Do the lines $\mathbf{r} = \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mu(3\mathbf{j} + 5\mathbf{k})$ intersect or are they skew? If they intersect, find the point of intersection.

Step 1
The direction vectors are not equal or multiples of each other, and hence the lines are not parallel. The parametric equations are:

$x = \lambda \qquad x = 2$
$y = -\lambda \qquad y = 1 + 3\mu$
$z = 1 - 3\lambda \qquad z = 5\mu$

Step 2
Equating values of *x*: $\lambda = 2$ equation (i)
Equating values of *y*: $-\lambda = 1 + 3\mu$ equation (ii)
Equating values of *z*: $1 - 3\lambda = 5\mu$ equation (iii)

Step 3
Solve equations (i) and (ii).
Substituting $\lambda = 2$ from equation (i) into equation (ii) gives $-2 = 1 + 3\mu$ $\Rightarrow \mu = -1$.

Step 4
Substitute $\lambda = 2$ and $\mu = -1$ into equation (iii).
$1 - 3(2) = 5(-1)$
$\Rightarrow -5 = -5$
Hence the lines intersect. Substituting either the value of λ or the value of μ into the equations will give the coordinates of the point of intersection.
Using $\mu = -1 \Rightarrow x = 2, y = 1 + 3(-1) = -2$ and $z = 5(-1) = -5$.
Hence the coordinates of the point of intersection are $(2, -2, -5)$.

Example

Do the lines $\mathbf{r} = -3\mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ intersect or are they skew? If they intersect find the point of intersection.

Step 1
The lines are not parallel. The parametric equations are:

$x = 2\lambda \qquad x = 1 + \mu$
$y = -3 + \lambda \qquad y = 2 - \mu$
$z = -2 + \lambda \qquad z = -5 + 2\mu$

Step 2
Equating values of *x*: $2\lambda = 1 + \mu$ equation (i)
Equating values of *y*: $-3 + \lambda = 2 - \mu$ equation (ii)
Equating values of *z*: $2 + \lambda = -5 + 2\mu$ equation (iii)

Step 3
Solve equations (i) and (ii).
$2\lambda - \mu = 1$ equation (i)
$\lambda + \mu = 5$ equation (ii)
Adding equations (i) and (ii) $\Rightarrow 3\lambda = 6 \Rightarrow \lambda = 2$
Substitute into equation (i) $\Rightarrow 2(2) = 1 + \mu \Rightarrow \mu = 3$

Step 4
Substitute $\lambda = 2$ and $\mu = 3$ into equation (iii).
$2 + 2 = -5 + 2(3) \Rightarrow 4 = 1$, which is not possible.
Hence the values of λ and μ do not fit equation (iii), and the lines are skew.

Example

a) Given that the lines $\mathbf{r}_1 = \begin{pmatrix}1\\-1\\3\end{pmatrix} + m\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix}2\\4\\6\end{pmatrix} + n\begin{pmatrix}2\\1\\3\end{pmatrix}$ intersect at point P, find the coordinates of P.
(b) Show that the point $A(3, -3, 5)$ and the point B(0, 3, 3) lie on $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively.
(c) Hence find the area of triangle APB.

a) **Step 1**
The direction vectors are not equal or multiples of each other and hence the lines are not parallel. The parametric equations are:

$x = 1 + m \qquad x = 2 + 2n$
$y = -1 - m \qquad y = 4 + n$
$z = 3 + m \qquad z = 6 + 3n$

Step 2
Equating values of *x*: $1 + m = 2 + 2n$ equation (i)
Equating values of *y*: $-1 - m = 4 + n$ equation (ii)
Equating values of *z*: $3 + m = 6 + 3n$ equation (iii)

Step 3
Solve equations (i) and (ii).
$m - 2n = 1$ equation (i)
$m + n = -5$ equation (ii)
Equation (i) – Equation (ii) $\Rightarrow 3n = -6 \Rightarrow n = -2$
Substitute into equation (i) $\Rightarrow m - 2(-2) = 1 \Rightarrow m = -3$

Step 4
In this case we do not need to prove that the lines intersect because we are told in the question. Substituting either the value of *n* or the value of *m* into the equations will give the coordinates of the point of intersection.
Using $n = -2 \Rightarrow x = 2 + 2(-2) = -2, y = 4 + (-2) = 2$ and $z = 6 + 3(-2) = 0$.
Hence the coordinates of P are $(-2, 2, 0)$.

b) If the point $(3, -3, 5)$ lies on $\mathbf{r}_1$ then there should be a consistent value for *m* in the parametric equations of $\mathbf{r}_1$.
$3 = 1 + m \Rightarrow m = 2$
If $m = 2, y = -1 - 2 = -3$ and $z = 3 + 2 = 5$. Hence A lies on $\mathbf{r}_1$.
If the point (0, 3, 3) lies on $\mathbf{r}_2$ then there should be a consistent value for *n* in the parametric equations of $\mathbf{r}_2$.

$0 = 2 + 2n \Rightarrow n = -1$

If $n = -1$, $y = 4 - 1 = 3$ and $z = 6 - 3 = 3$. Hence B lies on $\mathbf{r}_1$.

c) $\overrightarrow{AP} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix}$ and $\overrightarrow{BP} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix}$

$$\Rightarrow \overrightarrow{AP} \times \overrightarrow{BP} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 5 & -5 \\ -2 & -1 & -3 \end{vmatrix}$$

$$= \mathbf{i}[(-15) - 5] - \mathbf{j}[15 - 10] + \mathbf{k}[5 - (-10)]$$

$$= -20\mathbf{i} - 5\mathbf{j} + 15\mathbf{k}$$

Area of triangle ABP $= \frac{1}{2}|\overrightarrow{AP} \times \overrightarrow{BP}|$

$$= \frac{1}{2}\sqrt{(-20)^2 + (-5)^2 + 15^2} = \frac{\sqrt{650}}{2} = \frac{5\sqrt{26}}{2} \text{ units}^2$$

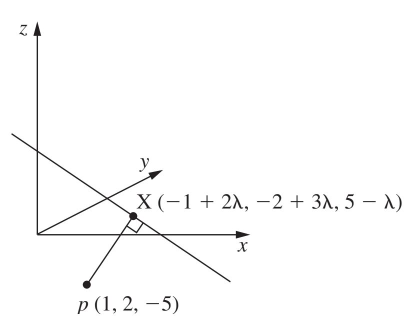
Example

Find the position vector of the point of intersection of the line $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ and the perpendicular that passes through the point P(1, 2, −5). Hence find the shortest distance from (1, 2, −5) to the line.

Any point on the line $\mathbf{r}$ is given by the parametric equations

$$x = -1 + 2\lambda$$
$$y = -2 + 3\lambda$$
$$z = 5 - \lambda$$

This could be point X as shown in the diagram.



Hence the direction vector $\overrightarrow{PX}$ is

$$(-1 + 2\lambda - 1)\mathbf{i} + (-2 + 3\lambda - 2)\mathbf{j} + (5 - \lambda - (-5))\mathbf{k}$$

$$= (-2 + 2\lambda)\mathbf{i} + (-4 + 3\lambda)\mathbf{j} + (10 - \lambda)\mathbf{k}.$$

If $\overrightarrow{PX}$ is perpendicular to $\mathbf{r}$ then the scalar product of $\overrightarrow{PX}$ and the direction vector of $\mathbf{r}$ must be zero.

$$\Rightarrow [(-2 + 2\lambda)\mathbf{i} + (-4 + 3\lambda)\mathbf{j} + (10 - \lambda)\mathbf{k}] \cdot [2\mathbf{i} - 3\mathbf{j} + \mathbf{k}] = 0$$
$$\Rightarrow -4 + 4\lambda + 12 - 9\lambda + 10 - \lambda = 0$$
$$\Rightarrow 6\lambda = 18$$
$$\Rightarrow \lambda = 3$$

Substituting $\lambda = 3$ into the parametric equations will give the point of intersection.

Hence: $x = -1 + 6 = 5$

$y = -2 + 9 = 7$

$z = 5 - 3 = 2$

Therefore the position vector of the point is $5\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$.

The shortest distance is the distance between (5, 7, 2) and (1, 2, −5).

Using Pythagoras' theorem, distance

$$= \sqrt{(5 - 1)^2 + (7 - 2)^2 + (2 - (-5))^2}$$
$$= \sqrt{4^2 + 5^2 + 7^2} = \sqrt{90} = 3\sqrt{10}$$

Finding the angle between two lines

Here we are interested in the direction of the two vectors, and it is the angle between these two vectors that we require.

Method

1 Find the direction vector of each line.
2 Apply the scalar product rule and hence find the angle.

Example

Find the angle θ between the lines

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

Step 1

The required direction vectors are $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

Step 2

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \sqrt{2^2 + 1^2 + (-1)^2}\sqrt{3^2 + (-2)^2 + 1^2} \cos\theta$$

$$\Rightarrow (2)(3) + (1)(-2) + (-1)(1) = \sqrt{6}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{3}{\sqrt{6}\sqrt{14}}$$

$$\Rightarrow \theta = 70.9°$$

Exercise 2

1 Determine whether the following pairs of lines are parallel, coincident, skew or intersecting. If they intersect, give the position vector of the point of intersection.

a $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -3 \\ 7 \\ -6 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix}1\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}0\\0\\5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\1\end{pmatrix}$

c $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + m(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and
$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + n(-8\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$

d $\mathbf{r} = \begin{pmatrix}1\\5\\-1\end{pmatrix} + s\begin{pmatrix}4\\-1\\2\end{pmatrix}$ and $x = 1 + 2t, y = 3 - 3t, z = 2t$

e $x = 2 + 3\lambda$ $x = 4 - 2\mu$
$y = 1 + \lambda$ and $y = 2 + 2\mu$
$z = -2 + 8\lambda$ $z = 6 + 16\mu$

f $x - 2 = \dfrac{y-3}{2} = \dfrac{z+1}{-3}$ and $-\dfrac{x}{2} = \dfrac{y+1}{-4} = \dfrac{z-5}{6}$

g $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + m(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\dfrac{3-x}{2} = \dfrac{2y+3}{2} = \dfrac{3z-5}{-2}$

2 Determine whether the given points lie on the given lines. If not, find the shortest distance from the point to the line.

a $(0, 0, 1)$ $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

b $(1, 2, -1)$ $\mathbf{r} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \mu\begin{pmatrix}2\\1\\2\end{pmatrix}$

c $(1, -1, 3)$ $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} + \mathbf{k})$

d $(-1, -2, 0)$ $\mathbf{r} = \begin{pmatrix}3\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\2\\-1\end{pmatrix}$

e $\left(4, 0, -\dfrac{7}{2}\right)$ $\dfrac{x+2}{3} = \dfrac{2-y}{1} = \dfrac{2z+3}{-2}$

f $(3, 2, 1)$ $x = 2 - m, y = 3m + 1, z = 1 - 2m$

3 The points $A(0, 1, -2)$, $B(3, 5, 5)$ and $D(1, 3, -3)$ are three vertices of a parallelogram ABCD. Find vector and Cartesian equations for the sides AB and AD and find the coordinates of C.

4 Find the acute angle between each of these pairs of lines.

a $\mathbf{r} = \begin{pmatrix}-2\\1\\-1\end{pmatrix} + s\begin{pmatrix}2\\6\\-1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}3\\-1\\-1\end{pmatrix} + t\begin{pmatrix}2\\-4\\-3\end{pmatrix}$

b $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + m(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$ and
$\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + n(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$

c $\mathbf{r} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + s\begin{pmatrix}2\\-3\\-4\end{pmatrix}$ and $x = 1 + t, y = 2 + 5t, z = 1 - t$

d $x = -1 + \lambda$ $x = 1 + \mu$
$y = 2 + 5\lambda$ and $y = 2 - 3\mu$
$z = -2 + 3\lambda$ $z = 3 + 5\mu$

e $\dfrac{x-3}{4} = \dfrac{y+2}{3} = \dfrac{z-2}{-3}$ and $-\dfrac{2x+1}{3} = \dfrac{3y+1}{5} = \dfrac{z-1}{3}$

5 Two lines have equations $\mathbf{r} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\2\\a\end{pmatrix}$.

Find the value of a for which the lines intersect and the position vector of the point of intersection.

6 a Show that the points whose position vectors are $\begin{pmatrix}9\\10\end{pmatrix}$ and $\begin{pmatrix}-3\\-8\end{pmatrix}$ lie on the line with equation $\mathbf{r} = \begin{pmatrix}1\\-2\end{pmatrix} + s\begin{pmatrix}2\\3\end{pmatrix}$.

b Obtain, in parametric form, an equation of the line that passes through the point with position vector $\begin{pmatrix}-3\\8\end{pmatrix}$ and is perpendicular to the given line.

7 The position vectors of the points A and B are given by $\overrightarrow{OA} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, where O is the origin.

a Find a vector equation of the straight line passing through A and B.

b Given that this line is perpendicular to the vector $p\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, find the value of p.

13.3 Equation of a plane

Definition of a plane

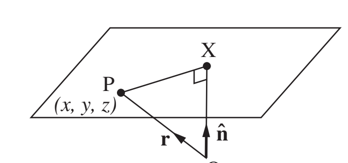
In simple terms a plane can be described as an infinite flat surface such that an infinite straight line joining any two points on it will lie entirely in it. This surface can be vertical, horizontal or sloping. The position of a plane can be described by giving

1. three non-collinear points
2. a point and two non-parallel lines
3. a vector perpendicular to the plane at a given distance from the origin
4. a vector perpendicular to the plane and a point that lies in the plane.

The vector equation of a plane is described as being a specific distance away from the origin and perpendicular to a given vector. The reason behind this is that it makes the vector equation of a plane very straightforward. We will first look at this form and then see how we can derive the other forms from this.

Scalar product form of the vector equation of a plane

Consider the plane below, distance d from the origin, which is perpendicular to the unit vector $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is directed away from O. X is the point of intersection of the perpendicular with the plane, P is any point (x, y, z), and $\overrightarrow{OP} = \mathbf{r}$.



$\overrightarrow{OX} = d\hat{\mathbf{n}}$

We also know that $\overrightarrow{OX}$ and $\overrightarrow{PX}$ are perpendicular and thus $\overrightarrow{OX} \cdot \overrightarrow{PX} = 0$.

Now $\overrightarrow{PX} = d\hat{\mathbf{n}} - \mathbf{r}$

Hence $d\hat{\mathbf{n}} \cdot (d\hat{\mathbf{n}} - \mathbf{r}) = 0$

$\Rightarrow d^2\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} - d\hat{\mathbf{n}} \cdot \mathbf{r} = 0$

$\Rightarrow d(\mathbf{r} \cdot \hat{\mathbf{n}}) = d^2$

Since $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ and scalar product is commutative

$\Rightarrow \mathbf{r} \cdot \hat{\mathbf{n}} = d$

$\mathbf{r} \cdot \hat{\mathbf{n}} = d$ is the standard equation of a plane, where $\mathbf{r}$ is the position vector of any point on the plane, $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane, and d is the distance of the plane from the origin.

Example

Write down the vector equation of the plane in scalar product form.

A plane is perpendicular to the vector $\frac{1}{5\sqrt{2}}(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ and 6 units away from the origin.

Since the perpendicular vector is a unit vector the vector equation of the plane is

$$\mathbf{r} \cdot \frac{1}{5\sqrt{2}}(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 6$$

or $\mathbf{r} \cdot (3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 30\sqrt{2}$

We can see that any equation of the form $\mathbf{r} \cdot \mathbf{n} = D$ represents a plane perpendicular to $\mathbf{n}$, known as the direction normal. If we want the plane in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d$, we divide by the magnitude of $\mathbf{n}$.

Example

Find the distance of the plane $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} = 8$ from the origin, and the unit vector perpendicular to the plane.

The magnitude of the direction vector is $\sqrt{3^2 + 2^2 + (-4)^2} = \sqrt{29}$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} \frac{3}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \\ \frac{-4}{\sqrt{29}} \end{pmatrix} = \frac{8}{\sqrt{29}}$$

Hence the distance from the origin is $\frac{8}{\sqrt{29}}$, and the unit vector perpendicular to the plane is $\begin{pmatrix} \frac{3}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \\ \frac{-4}{\sqrt{29}} \end{pmatrix}$.

It is not common to be told the distance of the plane from the origin; it is much more likely that a point on the plane will be given. In this case we can still use the scalar product form of the vector equation, as shown in the next example.

Example

A plane is perpendicular to the line $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and passes through the point $(-1, 2, 5)$. Find the vector equation of the plane in scalar product form.

The equation of the plane must be of the form $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = D$.

Also, the position vector of the point must fit the equation.

$\Rightarrow (-\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = D$

$\Rightarrow D = -1 - 2 - 10 = -13$

Therefore the vector equation of the plane is $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = -13$.

If we are required to find the distance of the plane from the origin or the unit vector perpendicular to the plane, we just divide by the magnitude of the direction normal.

In this case $\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $d = \frac{-13}{\sqrt{6}}$.

Cartesian equation of a plane

The Cartesian equation is of the form $ax + by + cz = d$ and is found by putting $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

If we have the equation $\mathbf{r} \cdot (2\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 6$, then $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (2\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 6$.

Therefore the Cartesian equation of the plane is $2x - 4y - z = 6$.

Example

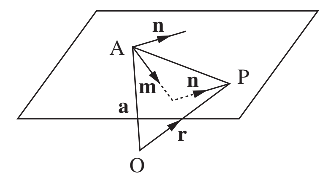
Convert $x + 3y - 2z = 9$ into scalar product form.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 9$$

Thus $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 9$

Parametric form of the equation of a plane

The parametric form is used to describe a plane passing through a point and containing two lines. Consider the diagram.



The vectors **m** and **n** are not parallel and lie on the plane. The point A, whose position vector is **a**, also lies in the plane.

Let P be any point on the plane with position vector **r**.

Hence $\overrightarrow{AP} = \lambda\mathbf{m} + \mu\mathbf{n}$, where λ and μ are parameters.

Thus $\mathbf{r} = \mathbf{a} + \overrightarrow{AP} = \mathbf{a} + \lambda\mathbf{m} + \mu\mathbf{n}$.

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{m} + \mu\mathbf{n}$, where **a** is the position vector of a point and **m** and **n** are direction vectors of lines, is the parametric form of the equation of a plane.

It is not easy to work in this form, so we need to be able to convert this to scalar product form.

In any plane, the direction normal is perpendicular to any lines in the plane. Hence to find the direction normal, we need a vector that is perpendicular to two lines in the plane. From Chapter 12, we remember that this is the definition of the vector product.

Method for converting the parametric form to scalar product form

1. Using the direction vectors of the lines, find the perpendicular vector using the vector product. This gives the plane in the form $\mathbf{r} \cdot \mathbf{n} = D$.
2. To find D, substitute in the coordinates of the point.

Example

Convert the parametric form of the vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$

to scalar product form.

In this equation the direction vectors of the lines are $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$.

Hence the direction normal of the plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = \mathbf{i}(0 - 3) - \mathbf{j}(0 + 2) + \mathbf{k}(-6 - 0)$$

$$= \begin{pmatrix} -3 \\ -2 \\ -6 \end{pmatrix}$$

Therefore the scalar product form of the vector equation of the plane is $\mathbf{r} \cdot \begin{pmatrix} -3 \\ -2 \\ -6 \end{pmatrix} = D$.

$\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ fits this $\Rightarrow \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -6 \end{pmatrix} = D$

$$\Rightarrow D = 3 + 2 - 12 = -7$$

Therefore the vector equation of the plane in scalar product form is $\mathbf{r} \cdot \begin{pmatrix} -3 \\ -2 \\ -6 \end{pmatrix} = -7$.

This could also be written in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d$:

$$\mathbf{r} \cdot \begin{pmatrix} \frac{-3}{7} \\ \frac{-2}{7} \\ \frac{-6}{7} \end{pmatrix} = -1$$

Example

Find the equation of the plane that contains the lines

$$\mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$$

$$\mathbf{r}_2 = -\mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

and passes through the point (2, 1, 3). Give the answer in scalar product form and in Cartesian form.

The direction vectors of the lines are $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Hence the direction normal of the plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i}(-2 - 4) - \mathbf{j}(3 - 4) + \mathbf{k}(3 + 2)$$

$$= -6\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

Therefore the vector equation of the plane is of the form $\mathbf{r} \cdot (-6\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = D$.

The vector $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ fits this. Hence $(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (-6\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = D$

$\Rightarrow D = -12 + 1 + 15 = 4$

Therefore the vector equation of the plane is $\mathbf{r} \cdot (-6\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = 4$.

This could also be written in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ as

$$\mathbf{r} \cdot \left(-\frac{6}{\sqrt{62}}\mathbf{i} + \frac{1}{\sqrt{62}}\mathbf{j} + \frac{5}{\sqrt{62}}\mathbf{k}\right) = \frac{4}{\sqrt{62}}$$

To find the Cartesian equation we use $\mathbf{r} \cdot (-6\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = 4$ and replace $\mathbf{r}$ with $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Hence $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (-6\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = 4$

$\Rightarrow -6x + y + 5z = 4$

Example

Find the scalar product form of the vector equation of the plane passing through the points A(1, −1, −3), B(2, −1, 1) and C(−3, −1, 2).

We can use the same method to do this, since we can form the direction vectors of two lines in the plane.

$\overrightarrow{AB} = (2 - 1)\mathbf{i} + (-1 + 1)\mathbf{j} + (1 + 3)\mathbf{k} = \mathbf{i} + 4\mathbf{k}$

$\overrightarrow{BC} = (-3 - 2)\mathbf{i} + (-1 + 1)\mathbf{j} + (2 - 1)\mathbf{k} = -5\mathbf{i} + \mathbf{k}$

Hence the direction normal of the plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ -5 & 0 & 1 \end{vmatrix} = \mathbf{i}(0 - 0) - \mathbf{j}(1 + 20) + \mathbf{k}(0 - 0) = -21\mathbf{j}$$

Therefore the vector equation of the plane is of the form $\mathbf{r} \cdot (-21\mathbf{j}) = D$.

$\overrightarrow{OA} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ fits this $\Rightarrow (\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \cdot (-21\mathbf{j}) = D$

$\Rightarrow D = 21$

Therefore the vector equation of the plane is $\mathbf{r} \cdot (-21\mathbf{j}) = 21$ or, in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d$, $\mathbf{r} \cdot \mathbf{j} = -1$.

Example

Find the equation of the plane that is parallel to the plane $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + m(\mathbf{i} + 2\mathbf{j}) + n(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ and passes through the point A(3, 2, −2). Hence find the distance of this plane from the origin.

The direction vectors of two lines parallel to the plane are $\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} - \mathbf{j} - 3\mathbf{k}$.

Hence the direction normal of the plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 1 & -1 & -3 \end{vmatrix} = \mathbf{i}(-6 - 0) - \mathbf{j}(-3 - 0) + \mathbf{k}(-1 - 2)$$

$$= -6\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

Therefore the scalar product form of the vector equation of the plane is $\mathbf{r} \cdot (-2\mathbf{i} + \mathbf{j} - \mathbf{k}) = D$.

$\overrightarrow{OA} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ fits this $\Rightarrow (3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \cdot (-2\mathbf{i} + \mathbf{j} - \mathbf{k}) = D$

$\Rightarrow D = -6 + 2 + 2 = -2$

Therefore the scalar product form of the vector equation of the plane is $\mathbf{r} \cdot (-2\mathbf{i} + \mathbf{j} - \mathbf{k}) = -2$.

In the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ this is $\mathbf{r} \cdot \left(-\frac{2}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) = \frac{-2}{\sqrt{6}}$.

The negative sign means that this plane is on the opposite side of the origin to the one where d is positive.

Hence the distance of this plane from the origin is $\left|\frac{-2}{\sqrt{6}}\right| = \frac{2}{\sqrt{6}}$.

Example

The plane π_1 has equation $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = 7$.

a) Find the distance of π_1 from the origin.
b) Find the equation of the plane π_2, which is parallel to π_1 and passes through the point A(−1, 2, 5).
c) Find and the distance of π_2 from the origin.
d) Hence find the distance between π_1 and π_2.

a) Putting π_1 in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d \Rightarrow \mathbf{r} \cdot \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{-1}{\sqrt{5}} \end{pmatrix} = \frac{7}{\sqrt{5}}$

Hence the distance of π_1 from the origin is $\frac{7}{\sqrt{5}}$.

b) Since π_1 and π_2 are parallel, they have the same direction normals.

Hence the equation of π_2 is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = D$.

$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$ fits this $\Rightarrow \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = D$

$\Rightarrow D = -2 - 5 = -7$

Therefore π_2 has equation $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = -7$.

c) We now put π_2 in the form $\mathbf{r}\cdot\hat{\mathbf{n}} = d \Rightarrow \mathbf{r}\cdot\begin{pmatrix}\frac{2}{\sqrt{5}}\\ 0\\ \frac{-1}{\sqrt{5}}\end{pmatrix} = \frac{-7}{\sqrt{5}}$

Hence the distance of π_2 from the origin is $\left|\frac{-7}{\sqrt{5}}\right| = \frac{7}{\sqrt{5}}$.

d) Hence the distance between π_1 and π_2 is $\frac{7}{\sqrt{5}} + \frac{7}{\sqrt{5}} = \frac{14}{\sqrt{5}}$.

We add the distances here because the planes are on opposite sides of the origin. Had they been on the same side of the origin, we would have subtracted them.

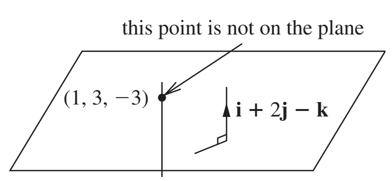
Lines and planes

Sometimes problems are given that refer to lines and planes. These are often solved by remembering that if a line and a plane are parallel then the direction vector of the line is perpendicular to the direction normal of the plane.

Example

Find the equation of the line that is perpendicular to the plane $\mathbf{r}\cdot(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 4$ and passes through the point $(1, 3, -3)$.

Since the line is perpendicular to the plane, the direction vector of the line is the direction normal to the plane. This is shown in the diagram.



Hence the vector equation of the line is $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

Example

Determine whether the line $\mathbf{r} = \begin{pmatrix}3\\-1\\0\end{pmatrix} + m\begin{pmatrix}2\\1\\4\end{pmatrix}$ intersects with, is parallel to, or is contained in the plane $\mathbf{r}\cdot\begin{pmatrix}5\\-2\\-2\end{pmatrix} = 17$.

We first test whether the direction vector of the line is perpendicular to the direction normal of the plane. If it is, then the line is either parallel to or contained in the plane. If not, it will intersect the plane.

We will deal with how to find this point of intersection later in the chapter.

Hence we find the scalar product of $\begin{pmatrix}2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}5\\-2\\-2\end{pmatrix} = 10 - 2 - 8 = 0$.

Since the scalar product is zero, the line is either parallel to the plane or contained in the plane. To find out which case this is, we find if they have a point in common. Knowing that the point $(3, -1, 0)$ lies on the line, we now test whether it lies in the plane.

$$\begin{pmatrix}3\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\-2\\-2\end{pmatrix} = 15 + 2 + 0 = 17$$

Hence the line is contained in the plane.

Exercise 3

1 Find the vector equation of each of these planes in the form $\mathbf{r}\cdot\hat{\mathbf{n}} = d$.

a Perpendicular to the vector $\begin{pmatrix}1\\4\\-1\end{pmatrix}$ and 3 units away from the origin

b Perpendicular to the vector $\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and containing the point $(3, 2, -1)$

c Perpendicular to the line $\mathbf{r} = \begin{pmatrix}1\\3\\-2\end{pmatrix} + m\begin{pmatrix}2\\0\\1\end{pmatrix}$ and 6 units away from the origin

d Perpendicular to the line $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + t(-\mathbf{i} - 4\mathbf{j} + \mathbf{k})$ and containing the point $(2, 3, -1)$

e Perpendicular to the line $\mathbf{r} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - 8\mathbf{j} + 3\mathbf{k})$ and passing through the point $A(3, 2, -2)$

f Containing the lines $\mathbf{r} = (\mathbf{i} + 2\mathbf{j}) + s(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}) + t(-\mathbf{i} + \mathbf{j} - 6\mathbf{k})$

g Passing through the points $A(1, 3, -2)$, $B(-6, 1, 0)$ and $C(-4, -3, -1)$

h Containing the lines $\mathbf{r} = \begin{pmatrix}-1\\-2\\\frac{11}{2}\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\-1\end{pmatrix}$ and $\frac{x-3}{4} = \frac{1-y}{2} = \frac{2z+3}{3}$

i Passing through the points A, B and C with position vectors $\overrightarrow{OA} = \begin{pmatrix}4\\1\\0\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}1\\0\\6\end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix}3\\-3\\4\end{pmatrix}$

j Passing through the origin and perpendicular to $\mathbf{r} = (2 - 3\mu)\mathbf{i} + (-3 - 4\mu)\mathbf{j} + (\mu - 6)\mathbf{k}$

k Passing through the point (5, 0, 5) and parallel to the plane $\mathbf{r}\cdot(3\mathbf{i} - 2\mathbf{j}) = 1$

l Passing through the origin and containing the line $\mathbf{r} = 3\mathbf{i} + \lambda(4\mathbf{j} + 7\mathbf{k})$

2 Convert these equations of planes into scalar product form.

a $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(2\mathbf{i}) + \mu(3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$

b $\mathbf{r} = (1 + 2s - 3t)\mathbf{i} + (2 - 5s)\mathbf{j} + (3 - 2s + t)\mathbf{k}$

c $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$

d $\mathbf{r} = (1 + 2m - n)\mathbf{i} + (3 - m - 4n)\mathbf{j} + (2 - m - 5n)\mathbf{k}$

3 Convert these equations to Cartesian form.

a $\mathbf{r}\cdot(\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) = 9$

b $\mathbf{r}\cdot\begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = -6$

c $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 5\mathbf{k}) + \lambda(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) + \mu(\mathbf{i} - 3\mathbf{j} - 3\mathbf{k})$

d $\mathbf{r} = (1 - 3s - 2t)\mathbf{i} + (3 + 4s)\mathbf{j} + (2 - 3s - t)\mathbf{k}$

4 Find the scalar product form of the vector equation of the plane that is perpendicular to the line $\mathbf{r}_1 = (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + s(\mathbf{i} - 3\mathbf{j} + 7\mathbf{k})$ and contains the line $\mathbf{r}_2 = (-3\mathbf{i} + \mathbf{k}) + s(2\mathbf{i} - 3\mathbf{j} - \mathbf{k})$.

5 Find the unit vectors perpendicular to these planes.

a $\mathbf{r}\cdot(\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 13$

b $4x - y + 5z = 8$

c $\mathbf{r} = (\mathbf{i} + 3\mathbf{j}) + m(2\mathbf{i} - \mathbf{j}) + n(4\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

d $5x = 2y - 4z + 15$

6 Show that the planes $\mathbf{r}\cdot(\mathbf{i} - 2\mathbf{j} + 15\mathbf{k}) = 19$ and $\mathbf{r}\cdot(3\mathbf{i} + 9\mathbf{j} + \mathbf{k}) = 9$ are perpendicular.

7 A plane passes through the points $(1, 1, -2)$ and $(3, 2, 0)$, and is parallel to the vectors $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Find the vector equation of the plane in scalar product form.

8 Two planes π_1 and π_2 have vector equations $\mathbf{r}\cdot(3\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 7$ and $\mathbf{r}\cdot(3\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 15$. Explain why π_1 and π_2 are parallel and hence find the distance between them.

9 **a** Show that the line L whose vector equation is $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ is parallel to the plane π_1 whose vector equation is $\mathbf{r}\cdot(\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 8$.

b Find the equation of plane π_2 that contains the line L and is parallel to π_1.

c Find the distance of π_1 and π_2 from the origin and hence determine the distance between the planes.

10 Find the equation of plane P_1 that contains the point with position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and is parallel to the plane P_2 with equation $\mathbf{r}\cdot(2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}) = 14$. Find the distances of P_1 and P_2 from the origin and hence determine the perpendicular distance between P_1 and P_2.

11 A plane passes through the point with position vector $\mathbf{i} + \mathbf{j}$ and is parallel to the lines $\mathbf{r}_1 = 2\mathbf{i} - 3\mathbf{j} + \lambda(\mathbf{i} + \mathbf{k})$ and $\mathbf{r}_2 = 5\mathbf{j} - \mathbf{k} + \mu(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. Find the vector equation of the plane in scalar product form. Is either of the given lines contained in the plane?

12 A plane passes through the three points whose position vectors are:

a $= \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

b $= \mathbf{i} - \mathbf{j} + 5\mathbf{k}$

c $= -4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

Find a vector equation of this plane in the form $\mathbf{r}\cdot\hat{\mathbf{n}} = d$ and hence write down the distance of the plane from the origin.

13 A plane passes through A, B and C, where $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 7 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$. Find the vector equation of the plane in scalar product form.

Hence find the distance of the plane from the origin.

14 Find a vector equation of the line through the point $(4, 3, 7)$ that is perpendicular to the plane $\mathbf{r}\cdot(2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 9$.

15 **a** Show that the line L with equation $x + 4 = y = \frac{z-5}{2}$ is parallel to the plane P_1 with equation $\mathbf{r}\cdot(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 8$.

b Find a vector equation of the plane P_2 in scalar product form that contains the line L and is parallel to P_1.

c Find the distances of P_1 and P_2 from the origin and hence find the distance between them.

16 Determine whether the given lines are parallel to, contained in, or intersect the plane $\mathbf{r}\cdot(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 5$.

a $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$

b $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j}) + s(3\mathbf{i} + 2\mathbf{k})$

c $\frac{x-1}{2} = \frac{2y+3}{4} = \frac{6z-1}{-12}$

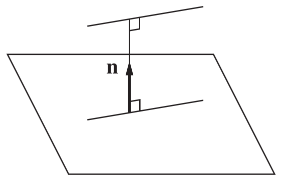
d $x = 2 - 3t,\ y = -t,\ z = 3t - 4$

13.4 Intersecting lines and planes

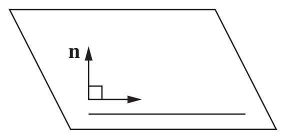
The intersection of a line and a plane

In this situation there are three possible cases.

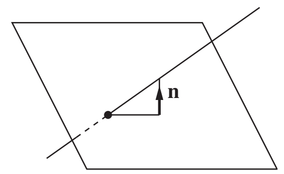
1. If the line is parallel to the plane there is no intersection.
2. If the line is contained in the plane, there are an infinite number of solutions, i.e. a line of solutions, which are given by the parametric equations of the line.
3. If the line intersects the plane, there is one solution.
   These are shown in the diagrams.



No solution



Infinite solutions



One solution

It is this third case we are interested in.

Method

1. Find an expression for any point on the line, which must fit the equation of the plane.
2. Find the value of the parameter, say λ.
3. Hence find the coordinates of the point of intersection.

Example

Find the point of intersection between the line $\mathbf{r} = (\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and the plane $\mathbf{r} \cdot (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 5$.

Any point on the line is given by the parametric equations of the line:

$$x = 1 + \lambda$$
$$y = 6 - \lambda$$
$$z = -5 + \lambda$$

Substituting these into the equation of the plane:

$$[(1+\lambda)\mathbf{i} + (6-\lambda)\mathbf{j} + (-5+\lambda)\mathbf{k}] \cdot [3\mathbf{i} + 2\mathbf{j} + \mathbf{k}] = 5$$
$$\Rightarrow 3(1+\lambda) + 2(6-\lambda) + 1(-5+\lambda) = 5$$
$$\Rightarrow 3 + 3\lambda + 12 - 2\lambda - 5 + \lambda = 5$$
$$\Rightarrow 2\lambda = -5$$
$$\Rightarrow \lambda = -\frac{5}{2}$$

Hence at the point of intersection:

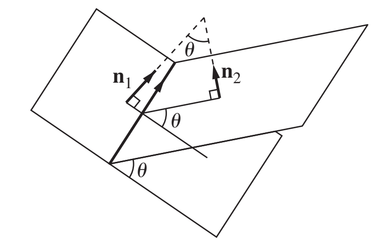
$$x = 1 - \frac{5}{2} = -\frac{3}{2}$$
$$y = 6 + \frac{5}{2} = \frac{17}{2}$$
$$z = -5 - \frac{5}{2} = -\frac{15}{2}$$

The point of intersection is $\left(-\frac{3}{2}, \frac{17}{2}, \frac{-15}{2}\right)$.

The intersection of two and three planes

This is directly related to the work done in Chapter 11, where the different cases were considered. Many of these questions are best solved using a method of inverse matrices or row reduction, but we will consider the situation of two planes intersecting in a line from a vector point of view.

Since the line of intersection of two planes is contained in both planes, it is perpendicular to both direction normals. This is shown in the diagram.



Method

1. Find the vector product of the direction normals. This gives the direction vector of the line.
2. Write the equations of the planes in Cartesian form.
3. We now assume that $z = 0$ since the line has to intersect this plane.
4. Solving simultaneously gives a point on the line.
5. Write down a vector equation of the line.

Example

Find a vector equation of the line of intersection of the planes $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 8$ and $\mathbf{r} \cdot (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 7$.

The direction normal of the plane is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 3 & -1 & 2 \end{vmatrix} = \mathbf{i}(2-3) - \mathbf{j}(4+9) + \mathbf{k}(-2-3) = -\mathbf{i} - 13\mathbf{j} - 5\mathbf{k}$$

The Cartesian equations of the plane are $2x + y - 3z = 8$ and $3x - y + 2z = 7$.

Assuming $z = 0$ gives $2x + y = 8$ and $3x - y = 7$.

$2x + y = 8$ equation (i)

$3x - y = 7$ equation (ii)

Equation (i) + equation (ii) $\Rightarrow x = 3$

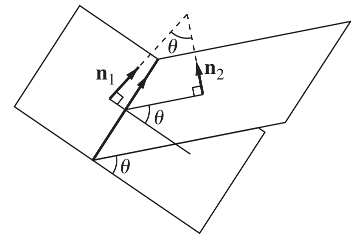
Substituting in equation (i) $\Rightarrow y = 2$

Thus a point on the line is (3, 2, 0).

Therefore an equation of the line of intersection is $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \lambda(-\mathbf{i} - 13\mathbf{j} - 5\mathbf{k})$.

Finding the angle between two planes

The angle between two planes is the same as the angle between the direction normals of the planes.



Method

Apply the scalar product to the direction normals of the planes and find the angle.

Example

Find the angle θ between the planes $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3$ and $\mathbf{r}\cdot(2\mathbf{i}+2\mathbf{j}-\mathbf{k}) = 1$.

$$(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+2\mathbf{j}-\mathbf{k}) = \sqrt{3}\sqrt{9}\cos\theta$$

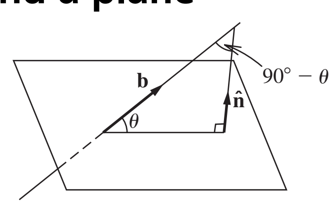
$$\Rightarrow 2+2-1 = 3\sqrt{3}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = 54.7°$$

Finding the angle between a line and a plane

The angle between a line and a plane is the complement of the angle between the line and the direction normal.



Method

1. Use the scalar product to find the angle between the direction normal of the plane and the direction vector of the line.
2. Subtract the angle in Step 1 from 90° to find the required angle.

Example

Find the angle between the line $\mathbf{r} = 3\mathbf{k} + \lambda(7\mathbf{i}-\mathbf{j}+4\mathbf{k})$ and the plane $\mathbf{r}\cdot(2\mathbf{i}-5\mathbf{j}-2\mathbf{k}) = 8$.

$$(7\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot(2\mathbf{i}-5\mathbf{j}-2\mathbf{k}) = \sqrt{66}\sqrt{33}\cos\theta$$

$$\Rightarrow 14+5-8 = \sqrt{66}\sqrt{33}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{13}{\sqrt{66}\sqrt{33}}$$

$$\Rightarrow \theta = 73.8°$$

Therefore the required angle is $90° - 73.8° = 16.2°$.

Exercise 4

1 Find the point of intersection between the line and plane.

a $x-3 = 2y+5 = 4-3z$ and $3x+2y-6z = 32$

b $\mathbf{r} = \begin{pmatrix}3\\1\\-2\end{pmatrix} + s\begin{pmatrix}2\\-1\\0\end{pmatrix}$ and $\mathbf{r}\cdot\begin{pmatrix}2\\2\\1\end{pmatrix} = 7$

c $\mathbf{r} = \mathbf{i} - 5\mathbf{k} + \mu(\mathbf{i}-\mathbf{j}-2\mathbf{k})$ and $3x-4y+z = 8$

d $x = 1+2m, y = m-3, z = 1$ and $\mathbf{r}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix} = 14$

e $\mathbf{r} = 4\mathbf{i}-\mathbf{j}+\lambda(6\mathbf{i}-\mathbf{j}+3\mathbf{k})$ and

$\mathbf{r} = 2\mathbf{k} + m(\mathbf{i}-4\mathbf{k}) + n(3\mathbf{i}-\mathbf{j}+6\mathbf{k})$

f $x = 4-3n, y = n-6, z = 1+2n$ and $x+y-2z = 20$

2 Find the acute angle between the two planes.

a $\mathbf{r}\cdot(3\mathbf{i}-4\mathbf{j}+9\mathbf{k}) = 10$ and $\mathbf{r}\cdot(3\mathbf{i}-6\mathbf{j}+2\mathbf{k}) = 15$

b $2x+3y-4z = 15$ and $x+y-4z = 17$

c $\mathbf{r}\cdot\begin{pmatrix}1\\0\\-3\end{pmatrix} = 1$ and $2x-y-5z = 6$

d $x-y-9z = 6$ and $y = 3x-z+5$

e $\mathbf{r}\cdot(\mathbf{i}-\mathbf{j}-5\mathbf{k}) = 2$ and

$\mathbf{r} = (1+m+n)\mathbf{i} + (2-m+3n)\mathbf{j} + (4-5m)\mathbf{k}$

f The plane perpendicular to the line $\mathbf{r} = 4\mathbf{i}-\mathbf{j}-5\mathbf{k}+s(\mathbf{i}-2\mathbf{j}+\mathbf{k})$ and passing though the point $(4, -1, -5)$, and the plane $\mathbf{r}\cdot(\mathbf{j}-5\mathbf{k}) = 14$

3 Find the acute angle between the line and plane.

a $\mathbf{r} = \mathbf{i}-2\mathbf{j}+\lambda(3\mathbf{i}+5\mathbf{j}+9\mathbf{k})$ and $\mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}+2\mathbf{k}) = 7$

b $\mathbf{r} = \begin{pmatrix}1\\2\\-3\end{pmatrix} + t\begin{pmatrix}3\\1\\-4\end{pmatrix}$ and $x+3y-4z = 5$

c $\dfrac{x-2}{5} = \dfrac{2y+7}{5} = \dfrac{3-2z}{4}$ and $2x-4y-9z = 0$

d $x = \lambda-1, y = \frac{1}{2}\lambda+3, z = 2+3\lambda$ and $\mathbf{r}\cdot(4\mathbf{i}-5\mathbf{k}) = -6$

e $\dfrac{x+4}{7} = 2y-1, z = -2$ and $\mathbf{r}\cdot\begin{pmatrix}4\\-5\\1\end{pmatrix} = 13$

f $\mathbf{r} = \mathbf{i}-2\mathbf{j}-\mathbf{k}+m(3\mathbf{i}-2\mathbf{j}+3\mathbf{k})$ and

$\mathbf{r} = 2\mathbf{i}-\mathbf{j}+4\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}+6\mathbf{k}) + \mu(2\mathbf{i}-\mathbf{j}-\mathbf{k})$

4 Find the vector equation of the line of intersection of the following pairs of planes.

a $\mathbf{r}\cdot(3\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = 8$ and $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}-5\mathbf{k}) = 7$

b $x - y - 2z = 4$ and $3x + y - 5z = 4$

c $\mathbf{r}\cdot(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = -2$ and $2x - y + 4z = 11$

d $\mathbf{r}\cdot(\mathbf{i} - \mathbf{j} + \mathbf{k}) = 7$ and $2x - y + 4z = 8$

e $\mathbf{r}\cdot(3\mathbf{i} - \mathbf{j} + 5\mathbf{k}) = 10$ and
$\mathbf{r} = (1 - 2m + n)\mathbf{i} + (3 - 4m + n)\mathbf{j} + (3m + 2n)\mathbf{k}$

f $\mathbf{r} = (1 - 2\lambda + 3\mu)\mathbf{i} + (2\lambda - \mu)\mathbf{j} + (6 - 5\mu)\mathbf{k}$ and
$\mathbf{r} = (2 - 3s)\mathbf{i} + (1 - 4t)\mathbf{j} + (s - 2t)\mathbf{k}$

5 Prove that the line $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} + \lambda(4\mathbf{i} - 5\mathbf{j} - 3\mathbf{k})$ is parallel to the intersection of the planes $\mathbf{r}\cdot(3\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 2$ and $\mathbf{r}\cdot(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 17$.

6 Find the acute angle between the plane defined by the points $(0, 1, -1)$, $(2, 1, 0)$ and $(3, -2, 0)$, and the plane defined by the points $(0, -1, 2)$, $(1, -1, 1)$ and $(-2, -3, 0)$.

7 a Find the equation of the straight line which passes through the point P$(1, 2, -3)$ and is perpendicular to the plane $3x + y - 3z = -5$.

b Calculate the coordinates of the point Q, which is the point of intersection of the line and the plane.

8 The vector equation of a plane is given by $\mathbf{r}\cdot(\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}) = 6$.

a Given that the point A$(a, 3a, 2a)$ lies on the plane, find the value of a.

b If B has coordinates $(1, 4, -1)$, find the acute angle between the direction normal of the plane and $\overrightarrow{AB}$.

c Hence find the perpendicular distance of B from the direction normal.

Review exercise

1 The points A, B, C, D have position vectors given by

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{k}$$
$$\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$
$$\mathbf{c} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$
$$\mathbf{d} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

respectively. Find

a a unit vector perpendicular to the plane ABC

b a vector equation, in the form $\mathbf{r}\cdot\hat{\mathbf{n}} = d$, of the plane parallel to ABC and passing through D

c the acute angle between the line BD and the perpendicular to the plane ABC.

2 The points A, B, C, D have position vectors given by

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$\mathbf{b} = \frac{3}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$
$$\mathbf{c} = 5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$
$$\mathbf{d} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

respectively. The point P lies on AB produced and is such that $\overrightarrow{AP} = 2\overrightarrow{AB}$, and the point Q is the midpoint of AC.

a Show that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AQ}$.

b Find the area of the triangle APQ.

c Find a unit vector perpendicular to the plane ABC.

d Find the equations of the lines AD and BD in Cartesian form.

e Find the acute angle between the lines AD and BD.

3 a The plane π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -3 \\ -9 \end{pmatrix}$.

The plane π_2 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

i For points that lie in π_1 and π_2, show that $\lambda = \mu$.

ii Hence, or otherwise, find a vector equation of the line of intersection of π_1 and π_2.

b The plane π_3 contains the line $\dfrac{2 - x}{3} = \dfrac{y}{-4} = z + 1$ and is perpendicular to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find the Cartesian equation of π_3.

c Find the intersection of π_1, π_2 and π_3. [IB May 05 P2 Q3]

4 The line L has equation

$$\mathbf{r} = \begin{pmatrix} 20 \\ 1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{where } \lambda \in \mathbb{R}$$

a Show that L lies in the plane whose equation is $\mathbf{r}\cdot\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -2$.

b Find the position vector of P, the foot of the perpendicular from the origin O to L.

c Find an equation of the plane containing the origin and L.

d Find the position vector of the point where L meets the plane π whose equation is $\mathbf{r}\cdot\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = 7$.

5 The line L has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$ and the plane π has equation $\mathbf{r}\cdot\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 5$.

a Find the coordinates of the point of intersection, P, of L and π.

b The point Q has coordinates (6, 1, 2), and R is the foot of the perpendicular from Q to π. Find the coordinates of R.

c Find a vector equation for the line PR.

d Find the angle PQR.

6 The point $A(2, 5, -1)$ is on the line L, which is perpendicular to the plane with equation $x + y + z - 1 = 0$.

a Find the Cartesian equation of the line L.

b Find the point of intersection of the line L and the plane.

c The point A is reflected in the plane. Find the coordinates of the image of the point A.

d Calculate the distance from the point B(2, 0, 6) to the line L. [IB Nov 03 P2 Q1]

7 Show that the planes $\mathbf{r}_1 \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 4$, $\mathbf{r}_2 \cdot (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 7$ and $\mathbf{r}_3 \cdot (\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) = 10$ intersect in a line and find the vector equation of the line.

8 Consider the points A(1, 2, 1), B(0, −1, 2), C(1, 0, 2) and D(2, −1, −6).

a Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

b Calculate $\overrightarrow{AB} \times \overrightarrow{BC}$.

c Hence, or otherwise, find the area of triangle ABC.

d Find the equation of the plane P containing the points A, B and C.

e Find a set of parametric equations for the line through the point D and perpendicular to the plane P.

f Find the distance from the point D to the plane P.

g Find a unit vector which is perpendicular to the plane P.

h The point E is a reflection of D in the plane P. Find the coordinates of E. [IB Nov 99 P2 Q2]

9 In a particular situation $\overrightarrow{OA} = 2\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{j} + 9\mathbf{k}$. The plane π is given by the equation $x + 3y + z = 1$.

a Determine whether or not A and B lie in the plane π.

b Find the Cartesian equation of the line AB.

c Find the angle between AB and the normal to the plane π at A.

d Hence find the perpendicular distance from B to this normal.

e Find the equation of the plane that contains AB and is perpendicular to π.

10 The equations of two lines L_1 and L_2 are

L_1: $\mathbf{r} = \mathbf{i} - 20\mathbf{j} - 13\mathbf{k} + t(7\mathbf{i} + \mathbf{k})$, where t is a scalar;

L_2: $\dfrac{x + 30}{2} = \dfrac{y + 39}{1} = \dfrac{2 - 3z}{2}$.

The equations of two planes P_1 and P_2 are

P_1: $6x + 3y - 2z = 12$;

P_2: $\mathbf{r} = 22\mathbf{i} + \lambda(\mathbf{i} + \mathbf{j}) + \mu(\mathbf{i} - 2\mathbf{k})$.

a Find the vector cross product $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - 2\mathbf{k})$.

b **i** Write down vectors $\mathbf{n}_1, \mathbf{n}_2$ that are normal to the planes P_1, P_2 respectively.

ii Hence, or otherwise, find the acute angle between the planes correct to the nearest tenth of a degree.

c Show that L_2 is normal to P_1.

d **i** Find the coordinates of the point of intersection of L_2 and P_1.

ii Hence, or otherwise, show that the two lines and the two planes all have a point in common. [IB Nov 98 P2 Q1]

11 A plane π_1 has equation $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j}) = -13$.

a Find, in vector form, an equation for the line passing through the point A with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and normal to the plane π_1.

b Find the position vector of the foot B of the perpendicular from A to the plane π_1.

c Find the sine of the angle between OB and the plane π_1.

The plane π_2 has equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 5$.

d Find the position vector of the point P where both the planes π_1, π_2 intersect with the plane parallel to the x-axis which passes through the origin.

e Find the position vector of the point Q where both the planes π_1, π_2 intersect with the plane parallel to the y-axis which passes through the origin.

f Find the vector equation of the line PQ.

12 The equations of the planes P_1 and P_2 are given by

P_1: $\mathbf{r} \cdot (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -1$

P_2: $\mathbf{r} \cdot (-2\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 4$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector for a point on the plane.

a Let L be the line of intersection of the two planes P_1 and P_2.

i Show that L is parallel to $3\mathbf{i} + 11\mathbf{j} + \mathbf{k}$.

ii Show that the point $A(0, -1, -1)$ lies on the line L. Hence, or otherwise, find the equation of L.

The equation of a third plane P_3 is given by

P_3: $\mathbf{r} \cdot (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) = c$.

b Determine the value of c for which the three planes, P_1, P_2 and P_3, intersect, and deduce whether this value of c gives a point of intersection or a line of intersection.

c For $c = 5$,

i show that the plane P_3 is parallel to the line L

ii find the distance between the line L and the plane P_3. [IB May 98 P2 Q3]

13 Using row reduction show that the following planes intersect in a line, and find the vector equation of the line of intersection.

$$x + 2y + z = 3$$
$$2x - y + 3z = 2$$
$$3x - 4y + 5z = 1$$

14 **a** The line L_1 is parallel to the vector $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and passes through the point (2, 3, 7). Find a vector equation of the line.

b The equation of a plane, E, is given by $2x + 3y - 4z + 21 = 0$. Find the point of intersection of the line L_1 and the plane E.

c Find an equation of a plane which passes through the point (1, 2, 3) and is parallel to the plane E.

d The parametric equations of another line L_2 are $x = t,\quad y = t$ and $z = -t,\ -\infty < t < \infty$.

Show that

i L_1 is not parallel to L_2

ii L_1 does not intersect L_2.

e Let O be the origin and P be the point.

i Find a vector $\mathbf{w}$ that is parallel to the line L_2.

ii Find the vector $\overrightarrow{PO}$.

iii Find the shortest distance d between the lines L_1 and L_2 by using the formula $d = \left|\dfrac{\overrightarrow{PO} \cdot (\mathbf{v} \times \mathbf{w})}{|\mathbf{v} \times \mathbf{w}|}\right|$. [IB May 97 P2 Q3]

15 **a** Show that the line L whose vector equation is

$$\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$$

is parallel to the plane P whose vector equation is $\mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = 4$.

b What is the distance from the origin to the plane?

c Find, in the same form as the equation given above for P, the equation of the plane P_1 which contains L and is parallel to the plane P.

d Deduce that the plane P_1 is on the opposite side of the origin to the plane P. Hence, or otherwise, find the distance between the line L and the plane P.

e Show that the plane P_1 contains the line whose vector equation is $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \gamma(2\mathbf{i} - \mathbf{j} + \mathbf{k})$. [IB May 93 P2 Q3]

16 **a** Show that the lines given by the parametric equations
$x = 3 + 4m,\ y = 3 - 2m,\ z = 7 - 2m$ and
$x = 7 + 2n,\ y = 1 - n,\ z = 8 + n$
intersect and find the coordinates of P, the point of intersection.

b Find the Cartesian equation of the plane π that contains these two lines.

c Find the coordinates of Q, the point of intersection of the plane and the line

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

d Find the coordinates of the point R if $|\overrightarrow{PR}| = |\overrightarrow{QR}| = 4$ and the plane of the triangle PQR is normal to the plane π.