16 Integration 3 – Applications

When students study integral calculus, the temptation is to see it as a theoretical subject. However, this is not the case. Pelageia Yakovlevna Polubarinova Kochina, who was born on 13 May 1899 in Astrakhan, Russia, spent much of her life working on practical applications of differential equations. Her field of study was fluid dynamics and *An application of the theory of linear differential equations to some problems of ground-water motion* is an example of her work. She graduated from the University of Petrograd in 1921 with a degree in pure mathematics. Following her marriage in 1925, Kochina had two daughters, Ira and Nina, and for this reason she resigned her position at the Main Geophysical Laboratory. However for the next ten years she continued to be active in her research and in 1934 she returned to full-time work after being given the position of professor at Leningrad University. In 1935 the family moved to Moscow and Kochina gave up her teaching position to concentrate on full-time research. She continued to publish until 1999, a remarkable achievement given that she was 100 years old!

16.1 Differential equations

An equation which relates two variables and contains a differential coefficient is called a differential equation. Differential coefficients are terms such as $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^ny}{dx^n}$.

The order of a differential equation is the highest differential coefficient in the equation. Therefore, a **first order equation** contains $\frac{dy}{dx}$ only. For example $\frac{dy}{dx} + 5y = 0$. However, a **second order equation** contains $\frac{d^2y}{dx^2}$ and could also contain $\frac{dy}{dx}$. An example of this would be $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 7y = 0$. Hence a differential equation of nth order would contain $\frac{d^ny}{dx^n}$ and possibly other lower orders.

A **linear differential equation** is one in which none of the differential coefficients are raised to a power other than one. Hence $x^2 + 5\left(\frac{dy}{dx}\right)^2 - 6y = 0$ is **not** a linear differential equation. Within the HL syllabus only questions on linear differential equations will be asked.

The solution to a differential equation has no differential coefficients within it. So to solve differential equations integration is needed. Now if $y = x^3 + k$, then $\frac{dy}{dx} = 3x^2$. $\frac{dy}{dx} = 3x^2$ is called the differential equation and $y = x^3 + k$ is called the solution.

Given that many things in the scientific world are dependent on rate of change it should come as no surprise that differential equations are very common and so the need to be able to solve them is very important. For example, one of the first researchers into population dynamics was Thomas Malthus, a religious minister at Cambridge University, who was born in 1766. His idea was that the rate at which a population grows is directly proportional to its current size. If t is used to represent the time that has passed since the beginning of the "experiment", then $t = 0$ would represent some reference time such as the year of the first census and p could be used to represent the population's size at time t. He found that $\frac{dp}{dt} = kp$ and this is the differential equation that was used as the starting point for his research.

http://calculuslab.sjdccd.cc.ca.us/ODE/7-A-3/7-A-3-h.html

Accessed 2 October 2005

A further example comes from physics. Simple harmonic motion refers to the periodic sinusoidal oscillation of an object or quantity. For example, a pendulum executes simple harmonic motion. Mathematically, simple harmonic motion is defined as the motion executed by any quantity obeying the differential equation $\frac{d^2x}{dt^2} = -\omega^2 x$.

Types of solution to differential equations

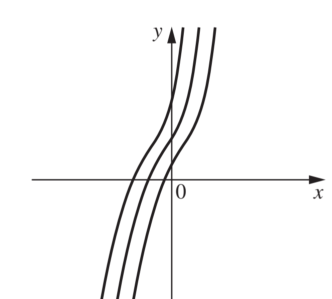
Consider the differential equation $\frac{dy}{dx} = e^{2x} - 4x$.

This can be solved using basic integration to give:

$$y = \frac{1}{2}e^{2x} - 2x^2 + k$$

There are two possible types of answer.

1. The answer above gives a family of curves, which vary according to the value which k takes. This is known as the **general solution**.



2. Finding the constant of integration, k, produces one specific curve, which is known as the **particular solution**. The information needed to find a particular solution is called the initial condition. For the example above, if we are told that (0, 5) lies on the curve, then we could evaluate k and hence find the particular solution.

$$y = \frac{1}{2}e^{2x} - 2x^2 + k$$

$$\Rightarrow 5 = \frac{1}{2} - 0 + k$$

$$\Rightarrow k = \frac{9}{2}$$

This is not the final answer.

$$y = \frac{1}{2}e^{2x} - 2x^2 + \frac{9}{2}$$

The final answer should always be in this form.

Always give the answer to a differential equation in the form $y = f(x)$, if possible.

If the general solution is required, the answer will involve a constant.

If the initial condition is given, then the constant should be evaluated and the particular solution given.

16.2 Solving differential equations by direct integration

Differential equations of the form $\frac{d^ny}{dx^n} = f(x)$ can be solved by integrating both sides.

If we are asked to solve $\frac{dy}{dx} = \frac{1}{1+x^2} + x\ln x$, then we can integrate to get

$$y = \int \frac{x}{1+x^2}dx + \int x\ln x\,dx.$$

It was shown in Chapter 15 that the first integral could be found using direct reverse and the second can be solved using the technique of integration by parts.

For the first integral:

We begin with $y = \ln(1+x^2)$

Hence $\frac{dy}{dx} = \frac{2x}{1+x^2}$

And therefore $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + k_1$

For the second integral:

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\left(\frac{1}{x}\right)dx$$

using integration by parts

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + k_2$$

Hence the solution to $\frac{dy}{dx} = \frac{1}{1+x^2} + x\ln x$ is $y = \frac{1}{2}\ln(1+x^2) + \frac{x^2}{2}\ln x - \frac{x^2}{4} + k.$

The two constants of integration k_1 and k_2 can be combined into one constant k.

If the values of y and x are given then k can be calculated. Given the initial condition that $y = 0$ when $x = 1$ we find that:

$$0 = \frac{1}{2}\ln 2 + 0 - \frac{1}{4} + k$$

$$\Rightarrow k = \frac{1}{4} - \frac{1}{2}\ln 2$$

$$y = \frac{1}{2}\ln(1+x^2) + \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{1}{4} - \frac{1}{2}\ln 2$$

This is the particular solution.

Many questions involving differential equations are set in a real-world context as many natural situations can be modelled using differential equations.

Example

The rate of change of the volume (V) of a cone as it is filled with water is directly proportional to the natural logarithm of the time (t) it takes to fill. Given that $\frac{dV}{dt} = \frac{1}{2}\text{cm}^3/\text{s}$ when $t = 5$ seconds and that $V = 25\text{ cm}^3$ when $t = 8$ seconds, find the formula for the volume.

We start with $\frac{dV}{dt} \propto \ln t$.

To turn a proportion sign into an equals sign we include a constant of proportionality, say k, which then needs to be evaluated.

Hence $\frac{dV}{dt} = k\ln t$

Given that $\frac{dV}{dt} = \frac{1}{2}$ when $t = 5$ we get:

$$\frac{1}{2} = k\ln 5$$

$$\Rightarrow k = \frac{1}{2\ln 5} = 0.311\ldots$$

So $\frac{dV}{dt} = 0.311\ln t$

$$\Rightarrow V = \int 0.311\ln t\,dt$$

$$\Rightarrow V = 0.311\int \ln t\,dt$$

Hence $V = 0.311\left[t\ln t - \int t\left(\frac{1}{t}\right)dt\right]$ using integration by parts

$$\Rightarrow V = 0.311[t\ln t - t + c]$$

The constant can be included within the brackets or it can be outside. It will evaluate to the same number finally.

Given that $V = 25$ when $t = 8$

$$25 = 0.311[8\ln 8 - 8 + c]$$

$$\Rightarrow c = 71.8$$

Hence $V = 0.311[t\ln t - t + 71.8]$

or $V = 0.311t\ln t - 0.311t + 22.3$

Given that the question is dealing with volume and time, this formula is only valid for $t > 0$.

In certain situations we may be asked to solve differential equations other than first order.

Example

Solve the differential equation $\frac{d^4y}{dx^4} = \cos x$, giving the general solution.

From basic integration:

$$\frac{d^3y}{dx^3} = \int \cos x \, dx$$

$$\Rightarrow \frac{d^3y}{dx^3} = \sin x + k$$

Continuing to integrate:

$$\frac{d^2y}{dx^2} = -\cos x + kx + c$$

$$\frac{dy}{dx} = -\sin x + \frac{kx^2}{2} + cx + d$$

$$y = \cos x + \frac{kx^3}{6} + \frac{cx^2}{2} + dx + e$$

If we were given the boundary conditions then the constants k, c, d, and e could be evaluated.

Example

Find the particular solution to the differential equation $\frac{d^3y}{dx^3} = 25e^{-5x} + 24x$ given that when $x = -1$, $\frac{d^2y}{dx^2} = -5e^5$, when $x = 1$, $\frac{dy}{dx} = -8$, and when $x = 0$, $y = 0$.

$$\frac{d^3y}{dx^3} = 25e^{-5x} + 24x$$

Hence $\frac{d^2y}{dx^2} = \int (25e^{-5x} + 24x) \, dx$

$$\Rightarrow \frac{d^2y}{dx^2} = -5e^{-5x} + 12x^2 + c$$

When $x = -1$, $\frac{d^2y}{dx^2} = -5e^5$

$$\Rightarrow -5e^{-5} = -5e^{-5} + 12 + c$$

$$\Rightarrow c = -12$$

$$\Rightarrow \frac{d^2y}{dx^2} = -5e^{-5x} + 12x^2 - 12$$

Integrating again gives:

$$\frac{dy}{dx} = \int (-5e^{-5x} + 12x^2 - 12) \, dx$$

$$\Rightarrow \frac{dy}{dx} = e^{-5x} + 4x^3 - 12x + d$$

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Now when $x = 1$, $\frac{dy}{dx} = -8$

$$\Rightarrow -8 = e^{-5} + 4 - 12 + d$$

$$\Rightarrow d = -e^{-5}$$

So $\frac{dy}{dx} = e^{-5x} + 4x^3 - 12x - e^{-5}$

The final integration gives:

$$y = \int (e^{-5x} + 4x^3 - 12x - e^{-5}) \, dx$$

$$\Rightarrow y = \frac{e^{-5x}}{-5} + x^4 - 6x^2 - e^{-5}x + f$$

When $x = 0$, $y = 0$ gives:

$$0 = -\frac{1}{5} + f$$

$$\Rightarrow f = \frac{1}{5}$$

Therefore $y = \frac{e^{-5x}}{-5} + x^4 - 6x^2 - e^{-5}x + \frac{1}{5}$

Exercise 1

Find the general solutions of these differential equations.

1 $\frac{dy}{dx} = x^2 + \sin x$ **2** $\frac{dy}{dx} = (3x - 7)^4$ **3** $\frac{dy}{dx} = 2x(1 - x^2)^{\frac{1}{2}}$

4 $\frac{dy}{dx} = x \sin x$ **5** $\frac{dy}{dx} = \frac{\cos x}{1 - \sin x}$ **6** $\frac{dy}{dx} = xe^{\frac{2x}{3}}$

7 $\frac{dy}{dx} = \frac{5x}{\sqrt{1 - 15x^2}}$ **8** $\frac{dy}{dx} = \sin^2 2x$ **9** $\frac{d^2y}{dx^2} = (3x + 2)^{\frac{1}{2}}$

10 $\frac{d^2y}{dx^2} = \sec^2 x$ **11** $\frac{d^3y}{dx^3} = x \ln x$ **12** $\frac{d^4y}{dx^4} = x \cos x$

Find the particular solutions of these differential equations.

13 $\frac{dy}{dx} = \frac{4x}{4x^2 + 3}$ given that when $x = 2$, $y = 0$

14 $\frac{dy}{dx} = 3 \sin\left(4x - \frac{\pi}{2}\right)$ given that when $x = \frac{\pi}{4}$, $y = 2$

15 $\frac{d^2y}{dx^2} = (2x - 1)^4$ given that when $x = \frac{1}{2}$, $\frac{dy}{dx} = 2$ and that when $x = 1$, $y = 4$

16 $\frac{d^2y}{dx^2} = \frac{2}{1 + x^2}$ given that when $x = \frac{\pi}{4}$, $\frac{dy}{dx} = 3$ and that when $x = 0$, $y = 5$

16.3 Solving differential equations by separating variables

Equations in which the variables are separable can be written in the form $g(y)\frac{dy}{dx} = f(x)$. These can then be solved by integrating both sides with respect to x.

Method

1. Put in the form $g(y)\frac{dy}{dx} = f(x)$

   This gives an equation in the form $\int g(y)\frac{dy}{dx}\,dx = \int f(x)\,dx$ which simplifies to $\int g(y)\,dy = \int f(x)\,dx$. The variables are now separated onto opposite sides of the equation.
2. Integrate both sides with respect to x.
3. Perform the integration.

Example

Find the general solution to the differential equation $\frac{dy}{dx} = \frac{4x-1}{2y}$.

Following step 1:

$$2y\frac{dy}{dx} = 4x - 1$$

Following step 2 and step 3:

$$\int 2y\frac{dy}{dx}\,dx = \int (4x-1)\,dx$$

$$\Rightarrow \int 2y\,dy = \int (4x-1)\,dx$$

$$\Rightarrow \frac{2y^2}{2} + k_1 = \frac{4x^2}{2} - x + k_2$$

$$\Rightarrow y^2 = 2x^2 - x + k$$

$$\Rightarrow y = \pm\sqrt{2x^2 - x + k}$$

Since there is an integral on each side of the equation, a constant of integration is theoretically needed on each side. For simplicity, these are usually combined and written as one constant.

Example

Solve to find the general solution of the equation $e^x\frac{dy}{dx} = \frac{x}{y^2+1}$.

Following step 1:

$$(y^2+1)\frac{dy}{dx} = xe^{-x}$$

Following step 2 and step 3:

$$\int (y^2+1)\frac{dy}{dx}\,dx = \int xe^{-x}\,dx$$

$$\Rightarrow \int (y^2+1)\,dy = \int xe^{-x}\,dx$$

$$\Rightarrow \frac{y^3}{3} + y = -xe^{-x} + \int 1e^{-x}\,dx$$

$$\Rightarrow \frac{y^3}{3} + y = -xe^{-x} - e^{-x} + k$$

In this situation an explicit equation in y cannot be found.

As was shown earlier, k can be evaluated if the initial condition that fits the equation is given. In this situation, we could be told that when $x = 2$, $y = 1$ and we can evaluate k.

$$\frac{1}{3} + 1 = 2e^{-2} + e^{-2} + k$$

$$\Rightarrow k = \frac{4}{3} - 3e^{-2}$$

$$\Rightarrow k = 0.927\ldots$$

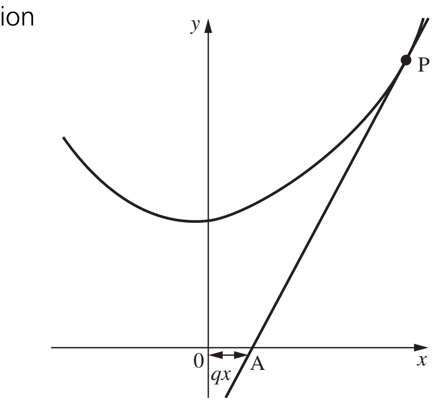
Therefore the final answer would be

$$\frac{y^3}{3} + y = xe^{-x} + e^{-x} + 0.927$$

Example

The diagram below shows a tangent to a curve at a point P which cuts the x-axis at the point A. Given that OA is of length qx, show that the points on the curve all satisfy the equation

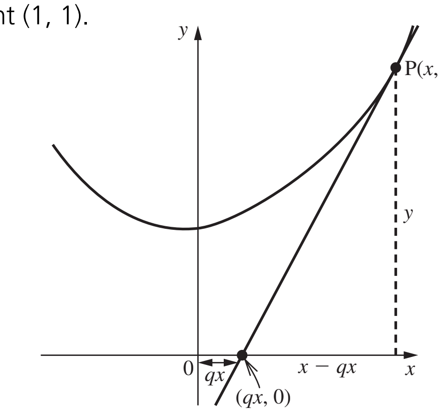
$$\frac{dy}{dx} = \frac{y}{x - qx}$$



a Hence show that the equation is of the form $y = kx^{\frac{1}{1-q}}$ where k is a constant.

b Given that q is equal to 2, find the equation of the specific curve which passes through the point (1, 1).

The gradient is $\frac{\Delta y}{\Delta x}$.



Therefore the gradient is $\frac{y-0}{x-qx}$.

Hence $\frac{dy}{dx} = \frac{y}{x-qx}$

a Following the method of separating variables:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x - qx}$$

$$\Rightarrow \int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{x - qx}dx$$

$$\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x - qx}dx$$

$$\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x(1 - q)}dx$$

$$\Rightarrow \ln|y| = \frac{1}{1 - q}\ln|x| + c$$

$$\Rightarrow \ln|y| = \frac{1}{1 - q}\ln|x| + \ln k$$

To simplify equations of this type (i.e. where natural logarithms appear in all terms) it is often useful to let $c = \ln k$.

Now by using the laws of logarithms:

$$\ln|y| - \ln k = \frac{1}{1 - q}\ln|x|$$

$$\Rightarrow \ln\left|\frac{y}{k}\right| = \ln|x|^{\frac{1}{1-q}}$$

$$\Rightarrow \frac{y}{k} = x^{\frac{1}{1-q}}$$

$$\Rightarrow y = kx^{\frac{1}{1-q}}$$

Technically the absolute value signs should remain until the end, but in this situation they are usually ignored.

b The curve passes through the point (1, 1) and $q = 2$.

$$1 = k(1)^{\frac{1}{1-2}}$$

$$\Rightarrow 1 = k(1)^{-1}$$

$$\Rightarrow k = 1$$

$$\Rightarrow y = 1x^{-1}$$

$$\Rightarrow y = \frac{1}{x}$$

Another real-world application of differential equations comes from work done with kinematics.

Example

A body has an acceleration a, which is dependent on time t and velocity v and is linked by the equation

$a = v \sin kt$

Given that when $t = 0$ seconds, $v = 1\ \text{ms}^{-1}$ and when $t = 1$ second, $v = 2\ \text{ms}^{-1}$, and that k takes the smallest possible positive value, find the velocity of the body after 6 seconds.

From the work on kinematics, we know that acceleration is the rate of change of velocity with respect to time, i.e. $a = \frac{dv}{dt}$.

Therefore the equation can be rewritten as $\frac{dv}{dt} = v \sin kt$.

This can be solved by separating variables.

$$\frac{1}{v}\frac{dv}{dt} = \sin kt$$

$$\Rightarrow \int \frac{1}{v}\frac{dv}{dt}dt = \int \sin kt\, dt$$

$$\Rightarrow \int \frac{1}{v}dv = \int \sin kt\, dt$$

$$\Rightarrow \ln|v| = -\frac{1}{k}\cos kt + c$$

The problem here is that when we substitute values for v and t, there are still two unknown constants. This is why two conditions are given. Now when $t = 0$ seconds, $v = 1\ \text{ms}^{-1}$ gives:

$$\ln|1| = -\frac{1}{k}\cos 0 + c$$

$$\Rightarrow 0 = -\frac{1}{k} + c$$

$$\Rightarrow c = \frac{1}{k}$$

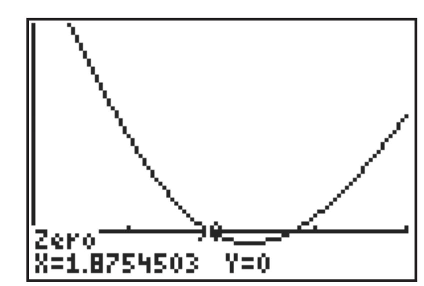
Hence $\ln|v| = -\frac{1}{k}\cos kt + \frac{1}{k}$

The other values can now be substituted to evaluate k:

$$\Rightarrow \ln 2 = -\frac{1}{k}\cos k + \frac{1}{k}$$

This equation cannot be solved by any direct means and so a graphing calculator needs to be used to find a value for k.

To do this, input the equation $y = \frac{1}{x}\cos x - \frac{1}{x} + \ln 2$ into a calculator and then solve it for $y = 0$. Since the question states that k should have the smallest possible positive value, then the value of k is the smallest positive root given by the calculator.



This value is $k = 1.88\ldots$ and hence $\frac{1}{k} = 0.533\ldots.$

The equation now reads $\ln|v| = 0.533 \cos 1.88t + 0.533$

The value of v when $t = 6$ seconds, can now be found by substituting in the value $t = 6$.

$$\ln|v| = 0.533 \cos(1.88 \times 6) + 0.533$$

$$\Rightarrow \ln|v| = 0.683\ldots$$

$$\Rightarrow v = e^{0.683}$$

$$\Rightarrow v = 1.98\ \text{ms}^{-1}$$

Exercise 2

Find the general solutions of these differential equations.

1 $y\frac{dy}{dx} = \tan x$ **2** $2x\frac{dy}{dx} = y^2 + 1$ **3** $\frac{dy}{dx} = \frac{3 + 2y}{4 - 3x}$

4 $(\sin x + \cos x)\frac{dy}{dx} = \cos x - \sin x$ **5** $\frac{y^3}{x}\frac{dy}{dx} = \ln x$ **6** $5x\frac{dy}{dx} = 6e^y$

7 $\frac{dy}{dx} = \frac{4y}{\sqrt{4 - x^2}}$ **8** $s^2\frac{ds}{dt} = \sin^{-1} t$ **9** $\frac{1}{x}\frac{dy}{dx} = \frac{(3x - 1)^9}{y^2}$

You will need to use the substitution $u = 3x - 1$ to perform the integration.

10 $v\frac{dv}{dt} = \cos^2 at$ **11** $e^{2x+y}\frac{dy}{dx} = 1$ **12** $3y(x + 1) = (x^2 + 2x)\frac{dy}{dx}$

Find the particular solutions of these differential equations.

13 $\frac{dy}{dx} = y(3 - x)^4$ given that when $x = 2, y = 4$

14 $e^{2x}\frac{dy}{dx} = \sqrt[3]{y}$ given that when $x = 1, y = 1$

15 $x\frac{dy}{dx} = \sin^2 y$ given that when $y = \frac{\pi}{4}, x = 4$

16 $\frac{2y}{3x}\frac{dy}{dx} = \frac{2y^2 + 3}{4x^2 - 1}$ given that when $x = 2, y = 8$

17 $\theta^2\frac{d\theta}{dt} = e^{2t}\sin t$ given that when $t = 0, \theta = \frac{\pi}{2}$

18 $\frac{ds}{dt} = \sqrt{t^2 - 9s^2t^2}$ given that when $t = 1, s = \frac{\pi}{3}$

The following exercise contains a mixture of questions on the material covered in this chapter so far.

Exercise 3

In questions 1 to 5, solve the differential equations.

1 $\frac{dy}{dx} = \tan x$ **2** $\frac{d^2x}{dt^2} - \omega^2 t = 0$ **3** $\frac{d^2x}{dt^2} = \omega \sin nt$

4 $\frac{1}{x^2}\frac{dy}{dx} = \frac{3}{y^2(1 + x^3)}$ given that when $x = 0, y = 3$ **5** $\cos^2 x\frac{dy}{dx} = \cos^2 y$

6 Consider the expression $z = x + y$.

a Using differentiation, find an expression for $\frac{dz}{dx}$.

b Hence show that the differential equation $\frac{dy}{dx} = (x + y)^2$ can be changed to $\frac{dz}{dx} = z^2 + 1$.

c Find the general solution of the differential equation $\frac{dz}{dx} = z^2 + 1$ and hence write down the solution to the differential equation $\frac{dy}{dx} = (x + y)^2$.

7 Find the particular solution to the equation $A\frac{d^4y}{dx^4} = B$, where A and B are constants that do not need to be evaluated, given that $y = 0$ and $\frac{d^2y}{dx^2} = 0$ for both $x = 0$ and $x = 1$.

8 A hollow cone is filled with water. The rate of increase of water with respect to time is $\frac{dV}{dt} = 4\sin\left(t + \frac{\pi}{4}\right)$. Given that when $t = \frac{\pi}{12}$ seconds, $V = 4\text{ cm}^3$, find a general formula for the volume V at any time t.

9 Oil is dripping out of a hole in the engine of a car, forming a thin circular film of the ground. The rate of increase of the radius of the circular film is given by the formula $\frac{dr}{dt} = 2\ln t^2$. Given that when $t = 5$ seconds, the radius of the film is 4 cm, find a general formula for the radius r at any time t.

10 The rate at which the height h of a tree increases is proportional to the difference between its present height and its final height s. Show that its present height is given by the formula $h = s - \frac{e^{-kt}}{B}$ where B is a constant.

11 Show that the equation of the curve which satisfies the differential equation $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$ and passes through the point $\left(\frac{\sqrt{3}}{3}, \sqrt{3}\right)$ is $y = (2 + \sqrt{3})x$.

16.4 Verifying that a particular solution fits a differential equation

The easiest way to tackle questions of this form is to differentiate the expression the required number of times and then substitute into the differential equation to show that it actually fits.

Example

Show that $y = 2e^{2x}$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 50e^{2x}$$

We begin with the expression $y = 2e^{2x}$.

Differentiating using the chain rule: $\frac{dy}{dx} = 4e^{2x}$

Differentiating again: $\dfrac{d^2y}{dx^2} = 8e^{2x}$

Substituting back into the left-hand side of the original differential equation gives:

$8e^{2x} + 6 \cdot 4e^{2x} + 9 \cdot 2e^{2x} = 50e^{2x}$

Since this is the same as the right-hand side, this is verified.

Sometimes the question will involve constants and in this case, on substitution, they will cancel out.

Example

Show that $y = Ae^{-2x} + Be^{-3x}$ is a solution to $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$.

We begin with the expression $y = Ae^{-2x} + Be^{-3x}$

Differentiating using the chain rule: $\dfrac{dy}{dx} = -2Ae^{-2x} - 3Be^{-3x}$

Differentiating again: $\dfrac{d^2y}{dx^2} = 4Ae^{-2x} + 9Be^{-3x}$

Substituting these back into the left-hand side of the original differential equation gives:

$$\begin{aligned}&4Ae^{-2x} + 9Be^{-3x} + 5(-2Ae^{-2x} - 3Be^{-3x}) + 6(Ae^{-2x} + Be^{-3x})\\ &= 4Ae^{-2x} + 9Be^{-3x} - 10Ae^{-2x} - 15Be^{-3x} + 6Ae^{-2x} + 6Be^{-3x}\\ &= 10Ae^{-2x} + 15e^{-3x} - 10Ae^{-2x} - 15Be^{-3x}\\ &= 0\end{aligned}$$

Since this is the same as the right-hand side, this is verified.

Exercise 4

Verify that these solutions fit the differential equations.

1 $y = 2x$, $\quad \dfrac{d^2y}{dx^2} + y = 2x$

2 $y = -\dfrac{1}{4}e^{-2x}$, $\quad \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 8y = 2e^{-2x}$

3 $y = e^x + \dfrac{1}{4}e^{2x} + \dfrac{1}{2}x + \dfrac{11}{4}$, $\quad \dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4 + x - \dfrac{1}{4}e^{2x}$

4 $y = e^x(A + Bx) + e^{2x}$, $\quad \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 3xe^x$

5 $y = e^x\left(A + Bx + \dfrac{1}{2}x^3\right)$, $\quad \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x$

6 $y = Ae^{-x} + Be^{-2x} + \dfrac{1}{10}(\sin x - 3\cos x)$, $\quad \dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sin x$

7 $y = Ae^{-\frac{x}{2}}\cos\dfrac{\sqrt{3}}{2}x + x$, $\quad \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 1 + x$

16.5 Displacement, velocity and acceleration

This is one of the more important applications of integral calculus. In Chapter 10, velocity and acceleration were represented as derivatives.

Reminder: if s is displacement, v is velocity and t is time, then:

$v = \dfrac{ds}{dt}$

and $a = \dfrac{d^2s}{dt^2} = \dfrac{dv}{dt}$.

Since we can represent velocity and acceleration using differential coefficients, solving problems involving velocity and acceleration often involves solving a differential equation.

Example

Given that the velocity v of a particle at time t is given by the formula $v = (t-1)^3$ and that when $t = 2$, $s = 6$, find the formula for the displacement at any time t.

Beginning with the formula: $v = (t-1)^3$

we know that $v = \dfrac{ds}{dt}$

So $\quad \dfrac{ds}{dt} = (t-1)^3$

Integrating both sides with respect to t gives:

$$\int \frac{ds}{dt}dt = \int (t-1)^3\,dt$$

$$\Rightarrow \int ds = \int (t-1)^3\,dt$$

$$\Rightarrow s = \frac{(t-1)^4}{4} + k$$

Now when $t = 2$, $s = 6$

$$6 = \frac{(2-1)^4}{4} + k$$

$$\Rightarrow k = \frac{23}{4}$$

$$\Rightarrow s = \frac{(t-1)^4}{4} + \frac{23}{4}$$

It is quite straightforward to find the displacement if there is a formula relating velocity and time and to find the velocity or the displacement if there is a formula relating acceleration and time. However, what happens when acceleration is related to displacement? The connection here was shown in Chapter 10 to be

$a = v\dfrac{dv}{ds}$.

Example

The acceleration of a particle is given by the formula $a = e^{2s}$. Given that when $s = 0, v = 2$, find the formula for the velocity in terms of the displacement s.

$$a = e^{2s}$$

$$\Rightarrow v\frac{dv}{ds} = e^{2s}$$

$$\Rightarrow \int v\frac{dv}{ds}ds = \int e^{2s}\,ds$$

$$\Rightarrow \int v\,dv = \int e^{2s}\,ds$$

$$\Rightarrow \frac{v^2}{2} = \frac{e^{2s}}{2} + k$$

We know that when $s = 0, v = 2$

$$\Rightarrow \frac{4}{2} = \frac{e^0}{2} + k$$

$$\Rightarrow k = \frac{3}{2}$$

Hence $\frac{v^2}{2} = \frac{e^{2s}}{2} + \frac{3}{2}$

$$\Rightarrow v^2 = e^{2s} + 3$$

$$\Rightarrow v = \pm\sqrt{e^{2s} + 3}$$

Example

A particle moves in a straight line with velocity v ms^{-1}. Its initial velocity is u ms^{-1}. At any time t, the velocity v is given by the equation $\frac{dv}{dt} + 3 + v^2 = 2$.

Prove that the particle comes instantaneously to rest after $\tan^{-1} u$ seconds. Given that the particle moves s metres in t seconds, show that the particle first comes to rest after a displacement of $\frac{1}{2}\ln|1 + u^2|$ metres.

$$\frac{dv}{dt} + 3 + v^2 = 2$$

$$\Rightarrow \frac{dv}{dt} = -(1 + v^2)$$

We solve this using the method of variables separable.

$$\Rightarrow \frac{1}{1+v^2}\frac{dv}{dt} = -1$$

$$\Rightarrow \int \frac{1}{1+v^2}\frac{dv}{dt}dt = -\int 1\,dt$$

$$\Rightarrow \tan^{-1} v = -t + k$$

It is given that when $t = 0, v = u$.

$$\Rightarrow \tan^{-1} u = -0 + k$$

$$\Rightarrow k = \tan^{-1} u$$

So $\tan^{-1} v = -t + \tan^{-1} u$

The particle comes to instantaneous rest when $v = 0$.

$$\Rightarrow \tan^{-1} 0 = -t + \tan^{-1} u$$

$$\Rightarrow t = \tan^{-1} u$$

Hence the result is proved.

To find the displacement s we use the fact that $\frac{dv}{dt} = v\frac{dv}{ds}$.

The equation $\frac{dv}{dt} = -(1 + v^2)$ becomes $v\frac{dv}{ds} = -(1 + v^2)$.

Again we solve this using the method of variables separable.

$$\frac{v}{1+v^2}\frac{dv}{ds} = -1$$

$$\Rightarrow \int \frac{v}{1+v^2}\frac{dv}{ds}ds = \int -1\,ds$$

$$\Rightarrow \int \frac{v}{1+v^2}dv = \int -1\,ds$$

To integrate the left-hand side, we use the method of direct reverse.

Letting $y = \ln(1 + v^2)$

$$\Rightarrow \frac{dy}{dv} = \frac{2v}{1+v^2}$$

$$\Rightarrow \int \frac{v}{1+v^2}dv = \frac{1}{2}\ln|1 + v^2| + k$$

So returning to the original equation:

$$\frac{1}{2}\ln|1 + v^2| = -s + c$$

When $v = u, s = 0$

$$\Rightarrow \frac{1}{2}\ln|1 + u^2| = -0 + c$$

$$\Rightarrow c = \frac{1}{2}\ln|1 + u^2|$$

So $\frac{1}{2}\ln|1 + v^2| = -s + \frac{1}{2}\ln|1 + u^2|$

We now find the displacement when the particle first comes to instantaneous rest, this is when $v = 0$.

$$\frac{1}{2}\ln|1 + 0^2| = -s + \frac{1}{2}\ln|1 + u^2|$$

$$\Rightarrow 0 = -s + \frac{1}{2}\ln|1 + u^2|$$

$$\Rightarrow s = \frac{1}{2}\ln|1 + u^2|$$

Hence the result is proved.

Exercise 5

1 The acceleration in ms^{-2} of a particle moving in a straight at time t is given by the formula $a = 4t^2 + 1$. When $t = 0, v = 0$ and $s = 0$. Find the velocity and displacement at any time t.

2 A particle starts to accelerate along a line AB with an initial velocity of $10\ ms^{-1}$ and an acceleration of $-6t^3$ at time t after leaving A.

a Find a general formula for the velocity v of the particle.
b Find the velocity after 8 seconds.
c Find a general formula for the displacement s of the particle.
d Find the displacement after 10 seconds.

3 The acceleration in ms^{-2} of a particle moving in a straight line at time t is given by the formula $a = 2t^3 + 3t - 4$. Given that the particle has an initial velocity of $6\ ms^{-1}$, find the distance travelled by the particle in the third second of its motion.

4 The acceleration in ms^{-2} of a particle moving in a straight line at time t is given by the formula $a = \cos 3t$. The particle is initially at rest when its displacement is 0.5 m from a fixed point O on the line.

a Find the velocity and displacement of the particle from O at any time t.
b Find the time that elapses before the particle comes to rest again.

5 The velocity of a particle is given by the formula $v = \frac{t^2 + 1}{1 - t}$ and is valid for all $t > 1$. Given that when $t = 2, s = 10$, find the displacement at any time t.

6 The acceleration of a particle is given by the formula $a = \sin\left(s + \frac{\pi}{4}\right)$. Given that when $s = \frac{\pi}{4}, v = 2$, find the formula for the velocity as a function of displacement for any s.

7 Consider a particle moving with acceleration $a = se^{2s}$. Given that when $s = 0$, $v = 2$, find the formula for the velocity as a function of displacement for any s.

8 A particle moves along a line AB. Given that A is 2 metres from O and it starts at A with a velocity of $2\ ms^{-1}$, find the formula for the velocity of the particle as a function of s, given that it has acceleration $a = (2s - 1)^4$.

9 The acceleration in ms^{-2} of a particle moving in a straight line at time t is given by the formula $a = -\frac{1}{t^2}$. When $t = 2$ seconds, $v = 8\ ms^{-1}$.

a Find the velocity when $t = 4$.
b Show that the particle has a terminal velocity of $7\frac{1}{2}\ ms^{-1}$.

10 Consider a particle with acceleration $e^{2t} - 4$. The particle moves along a straight line, PQ, starting from rest at P.

a Show that the greatest speed of the particle in its motion along PQ is $\left(\frac{3}{2} - \ln 16\right) ms^{-1}$.

b Find the distance covered by the particle in the first four seconds of its motion.

11 A bullet is decelerating at a rate of $kv\ ms^{-2}$ when its velocity is $v\ ms^{-1}$. During the first $\frac{1}{2}$ second the bullet's velocity is reduced from $220\ ms^{-1}$ to $60\ ms^{-1}$.

a Find the value of k.
b Deduce a formula for the velocity at any time t.
c Find the distance travelled during the time it takes for the velocity to reduce from $220\ ms^{-1}$ to $60\ ms^{-1}$.

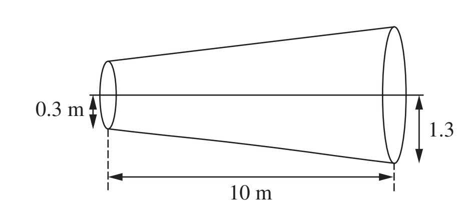
12 Consider a particle with acceleration $\sin \omega t\ ms^{-2}$. The particle starts from rest and moves in a straight line.

a Find the maximum velocity of the particle.
b The particle's motion is periodic. Give the time period of the particle. (This is the time taken between it achieving its maximum speeds.)

13 A particle is moving vertically downwards. Gravity is pulling it downwards, but there is a force of kv acting against gravity. Hence the acceleration experienced by the particle at time t is $g - kv$. If the particle starts from rest, find the velocity at any time t. Does this velocity have a limiting case?

16.6 Volumes of solids of revolution

Consider the problem of finding the volume of this tree trunk.



There are a number of ways that this can be done.

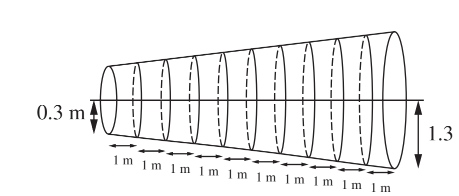
First, we could assume that the tree trunk is a cylinder of uniform radius and calculate the volume of the cylinder.

To do this, we need to calculate an average radius. This would be $\frac{1.3 + 0.3}{2} = 0.8$ m.

Hence the volume of the tree trunk is $\pi r^2 h = \pi \times 0.8^2 \times 10 = 6.4\pi\ m^3 = 20.1\ m^3$.

This is an inaccurate method.

A better way would be to divide the tree trunk into 10 equal portions as shown below and then calculate the volume of each portion.

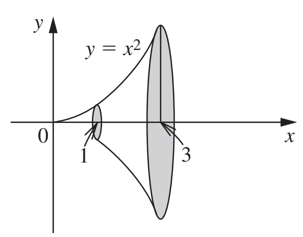


If we take an average radius for each section and assume that each portion is approximately a cylinder with height 1 m, then the volume will be:

$$(\pi \cdot 0.35^2 \cdot 1) + (\pi \cdot 0.45^2 \cdot 1) + (\pi \cdot 0.55^2 \cdot 1) + (\pi \cdot 0.65^2 \cdot 1) + (\pi \cdot 0.75^2 \cdot 1) +$$
$$(\pi \cdot 0.85^2 \cdot 1) + (\pi \cdot 0.95^2 \cdot 1) + (\pi \cdot 1.05^2 \cdot 1) + (\pi \cdot 1.15^2 \cdot 1) + (\pi \cdot 1.25^2 \cdot 1)$$
$$= \pi \cdot 1(0.35^2 + 0.45^2 + 0.55^2 + 0.65^2 + 0.75^2 + 0.85^2 + 0.95^2 + 1.05^2 +$$
$$1.15^2 + 1.25^2)$$
$$= 22.7 \text{ m}^3$$

This is still an approximation to the actual answer, but it is a better approximation than the first attempt. As we increase the number of portions that the tree trunk is split into, the better the accuracy becomes.

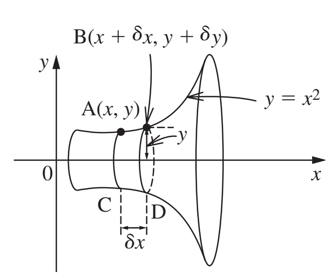
Consider the case of the curve $y = x^2$. If the part of the curve between $x = 1$ and $x = 3$ is rotated around the x-axis, then a volume is formed as shown below.



This is known as a volume of solid of revolution. The question now is how to calculate this volume. In Chapter 14, to find the area under the curve, the curve was split into infinitesimally thin rectangles and then summed using integration. Exactly the same principle is used here except rather than summing infinitesimally thin rectangles, we sum infinitesimally thin cylinders.

Effectively, this is what we did when we found the volume of the tree trunk.

Consider the diagram below.



Look at the element ABCD where A is on the curve and has coordinates (x, y). Since in this case $y = x^2$, then the coordinates of A are (x, x^2). ABCD is approximately a cylinder with radius y and whose "height" is δx.

Therefore the volume of $\text{ABCD} \approx \pi y^2\,\delta x$ and hence the volume V of the entire solid $\approx \sum_{x=a}^{x=b} \pi y^2\,\delta x$.

a and *b* are the boundary conditions which ensure that the volume is finite.

The smaller δx becomes, the closer this approximation is to V,

i.e. $V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2\,\delta x$.

$$V = \pi \int_a^b y^2\,dx$$

This is the formula for a full revolution about the x-axis.

In this case $y = x^2$,

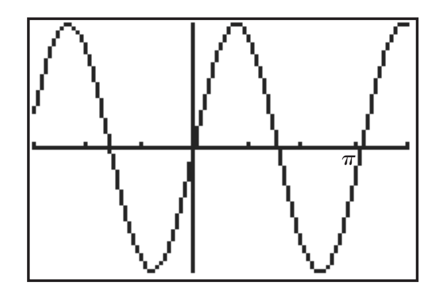
$$V = \pi \int_1^3 y^2\,dx$$
$$\Rightarrow V = \pi \int_1^3 x^4\,dx$$
$$\Rightarrow V = \pi \left[\frac{x^5}{5}\right]_1^3$$
$$\Rightarrow V = \pi \left[\frac{243}{5} - \frac{1}{5}\right]$$
$$\Rightarrow V = \frac{242\pi}{5}$$

If this question appeared on a calculator paper then the integration can be done on a calculator.

Example

Find the volume generated when one complete wavelength of the curve $y = \sin 2x$ is rotated through 2π radians about the x-axis.

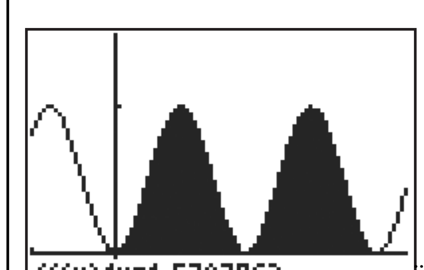
By drawing the curve on a graphing calculator it is evident that there are an infinite number of complete wavelengths. In this case the one which lies between 0 and π will be chosen.



The volume of the solid formed is given by the formula $V = \pi \int_0^{\pi} y^2\,dx$.

Hence $V = \pi \int_0^{\pi} \sin^2 2x\,dx$.

At this stage the decision on whether to use a graphing calculator or not will be based on whether the question appears on the calculator or non-calculator paper. The calculator display for this is shown below. In this case an answer of 4.93 units^3 is found.



This answer will need to be multiplied by π.

On a non-calculator paper we would proceed as follows.
From the trigonometrical identities we have:

$\cos 4x = \cos^2 2x - \sin^2 2x$

and $\cos^2 2x + \sin^2 2x = 1$,

giving $\cos^2 2x = 1 - \sin^2 2x$

So $\cos 4x = 1 - \sin^2 2x - \sin^2 2x$

$\Rightarrow \cos 4x = 1 - 2\sin^2 2x$

$\Rightarrow \sin^2 2x = \dfrac{1 - \cos 4x}{2}$

Hence

$$V = \pi \int_0^{\pi} \left(\frac{1 - \cos 4x}{2}\right) dx$$

$$\Rightarrow V = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 4x)\, dx$$

$$\Rightarrow V = \frac{\pi}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi}$$

$$\Rightarrow V = \frac{\pi}{2}\left[\left(\pi - \frac{\sin 4\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right]$$

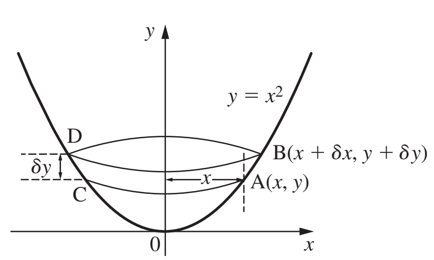
$$\Rightarrow V = \frac{\pi}{2}[(\pi - 0) - (0 - 0)]$$

$$\Rightarrow V = \frac{\pi^2}{2} \text{ units}^3$$

It is possible to rotate curves around a variety of different lines, but for the purposes of this syllabus it is only necessary to know how to find the volumes of solids of revolution formed when rotated about the *x*- or the *y*-axes.

Volumes of solids of revolution when rotated about the *y*-axis

The method is identical to finding the volume of the solid formed when rotating about the *x*-axis. Consider the curve $y = x^2$.



Look at the element ABCD where A is on the curve and has coordinates (x, y). If $y = x^2$, then $x = y^{\frac{1}{2}}$. Hence the coordinates of A are $(y^{\frac{1}{2}}, y)$. ABCD is approximately a cylinder with radius x and whose "height" is δy.

Therefore the volume of ABCD $\approx \pi x^2\, \delta y$

and the volume V of the entire solid $\approx \displaystyle\sum_{y=a}^{y=b} \pi x^2\, \delta y$.

The smaller δy becomes, the closer this approximation is to V,

i.e. $V = \displaystyle\lim_{\delta y \to 0} \sum_{y=a}^{y=b} \pi x^2\, \delta y$

$$V = \pi \int_a^b x^2\, dy$$

This is the formula for a full revolution about the *y*-axis.

In this case $x = y^{\frac{1}{2}}$. To find the volume of the solid formed when the part of the curve between $y = 0$ and $y = 2$ is rotated about the *y*-axis, we proceed as follows.

$$V = \pi \int_0^2 x^2\, dy$$

$$\Rightarrow V = \pi \int_0^2 y\, dy$$

$$\Rightarrow V = \pi \left[\frac{y^2}{2}\right]_0^2$$

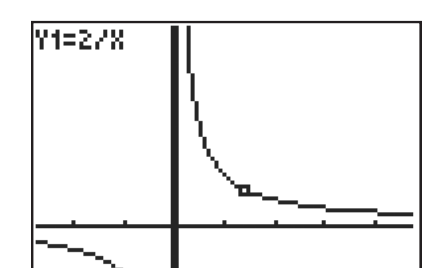
$$\Rightarrow V = \pi \left[\frac{4}{2} - 0\right]$$

$$\Rightarrow V = 2\pi$$

Example

Find the volume of the solid of revolution formed when the area bounded by the curve $xy = 2$ and the lines $x = 0$, $y = 3$, $y = 6$ is rotated about the *y*-axis.

As before, plotting the curve on a calculator or drawing a diagram first is a good idea.



As the rotation is taking place about the *y*-axis, the required formula is:

$$V = \pi \int_3^6 x^2\, dy$$

Since $xy = 2$, then $x = \dfrac{2}{y}$

So we have:

$$V = \pi \int_3^6 \left(\frac{2}{y}\right)^2 dy$$

$$\Rightarrow V = \pi \int_3^6 \frac{4}{y^2}\, dy$$

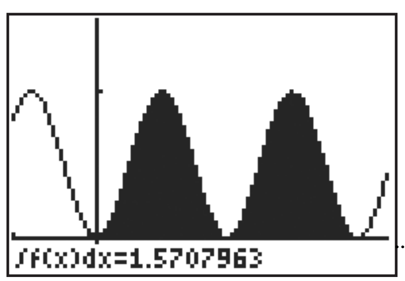
$$\Rightarrow V = \pi \int_3^6 4y^{-2}\, dy$$

$$\Rightarrow V = \pi \left[\frac{4y^{-1}}{-1}\right]_3^6$$

$$\Rightarrow V = \pi \left[\left(\frac{-4}{6}\right) - \left(\frac{-4}{3}\right)\right]$$

$$\Rightarrow V = \frac{2\pi}{3}$$

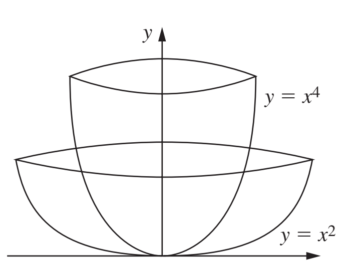
The calculator display is shown below.



Remember that this answer will need to be multiplied by π.

Hence $V = 2.09$ units^3.

Up until now it appears that volumes of solids of revolution are a theoretical application of integration. However, this is not the case. In the field of computer-aided design, volumes of solids of revolution are important. If, for example, we wanted to design a wine glass, then we could rotate the curve $y = x^2$ around the y-axis to give a possible shape. If we wanted a thinner wine glass, then we could rotate the curve $y = x^4$. Because this can be all modelled on a computer, and a three-dimensional graphic produced, designers can work out the shape that they want.



Exercise 6

1 Find the volumes generated when the following areas are rotated through 2π radians about the x-axis.

a The area bounded by the curve $y = 3x + 2$, the x-axis, the y-axis and the line $x = 2$.

b The area bounded by the curve $y = 4x - x^2$ and the x-axis.

c The area bounded by the curve $y = x^3$, the x-axis and the line $x = 2$.

d The area bounded by the curve $y = 1 + \sqrt{x}$, the x-axis, the y-axis and the line $x = 1$.

e The area bounded by the curve $y = x^2 - 1$, the x-axis and the line $x = 3$.

f The area bounded by the curve $y = \frac{1}{3}(2x - 1)^2$, the x-axis and the line $x = 5$.

g The area bounded by the curve $y = 9x - x^2 - 14$ and the x-axis.

h The area bounded by the curve $y = \sin 4x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{4}$.

i The area bounded by the curve $y = \tan 2x$, the x-axis and line $x = \frac{\pi}{6}$.

j The area bounded by the curve $y = e^{2x}\sin x$, the x-axis and the lines $x = 0.5$ and $x = 1.5$.

2 Find the volumes generated when the following areas are rotated through 2π radians about the y-axis.

a The area bounded by the curve $y = 4 - x^2$ and the x-axis.

b The area bounded by the curve $y = x^3$, the y-axis and the line $y = 2$.

c The area bounded by the curve $y = e^x$, the y-axis and the line $y = 2$.

d The area bounded by the curve $y = \sin x$, the y-axis and the lines $y = 0.2$ and $y = 0.8$.

e The area bounded by the curve $y = x^2 - 4x$, the y-axis, the x-axis and the line $y = 2$.

f The area bounded by the curve $y = \sin^{-1} x$, the y-axis and the line $y = \frac{\pi}{3}$.

g The area bounded by the curve $y = \ln(x + 1)$, the y-axis and the line $y = 1.5$.

3 Find a general formula for the volume generated when the area bounded by the curve $y = x^2$, the x axis and the line $x = a$ is rotated through 2π radians about the x-axis.

4 Find the volume obtained when the region bounded by the curve $y = 3 + \dfrac{4}{x}$, the x-axis and the lines $x = 3$ and $x = 6$ is rotated through 360° about the x-axis.

5 Consider the curve $y = x^{\frac{1}{3}}$. The part of the curve between $y = 2$ and $y = 5$ is rotated through 2π radians about the y-axis. Find the volume of the solid of revolution formed.

6 Find the volume generated when the area bounded by the curve $y = \ln x$, the x-axis, the y-axis and the line $y = 2$ is rotated through $360°$ about the y-axis.

7 Consider the curve $y = \frac{1}{5}x^2$. A volume is formed by revolving this curve through $360°$ about the y-axis. The radius of the rim of this volume is 5 cm. Find the depth of the shape and its volume.

8 Sketch the curve $y = |x^2 - 1|$ and shade the area that is bounded by the curve and the x-axis. This area is rotated through 2π radians about the x-axis. Find the volume generated. What is the volume when it is rotated through 2π radians about the y-axis?

9 The parabola $y = 8x^2$ is rotated through $360°$ about its axis of symmetry, thus forming a volume of solid of revolution. Calculate the volume enclosed between this surface and a plane perpendicular to the y-axis. This plane is a distance of 7 units from the origin.

10 Find the volume of the solid of revolution formed when the area included between the x-axis and one wavelength of the curve $y = b\sin\frac{x}{a}$ is rotated through $360°$ about the x-axis.

11 Consider the part curve $y^2 = x^2 \sin x$ which lies between $x = (n - 1)\pi$ and $x = (n + 1)\pi$ where n is an integer. Find the volume generated when this area is rotated through 2π radians about the x-axis.

12 **a** Using the substitution $x = \sin\theta$, evaluate $\int \sqrt{1 - x^2}\,dx$.

b Hence or otherwise, find the volume generated when the area bounded by the curve $y^2 = 1 - x^4$, the y-axis and the line $y = a$, $0 < a < 1$, is rotated through 2π radians about the y-axis.

Review exercise

1 The region A is bounded by the curve $y = \sin\left(2x + \frac{\pi}{3}\right)$ and by the lines $x = 0$ and $x = \frac{\pi}{6}$. Find the exact value of the volume formed when the area A is rotated fully about the x-axis.

2 Solve the differential equation $xy\frac{dy}{dx} = 1 + y^2$, given that $y = 0$ when $x = 2$.

[IB Nov 00 P1 Q17]

3 A particle moves in a straight line with velocity, in metres per second, at time t seconds, given by $v(t) = 6t^2 - 6t$, $t \geq 0$.

Calculate the total distance travelled by the particle in the first two seconds of motion. [IB Nov 02 P1 Q11]

4 Consider the region bounded by the curve $y = e^{-2x}$, the x-axis and the lines $x = \pm a$. Find in terms of a, the volume of the solid generated when this region is rotated through 2π radians about the x-axis.

5 Solve the differential equation $\frac{dy}{dx} = 5xy$ and sketch one of the solution curves which does not pass through $y = 0$.

6 The acceleration of a body is given in terms of the displacement s metres as $a = \frac{3s}{s^2 + 1}$. Determine a formula for the velocity as a function of the displacement given that when $s = 1$ m, $v = 2$ ms^{-1}. Hence find the exact velocity when the body has travelled 5 m.

7 The temperature T°C of an object in a room, after t minutes, satisfies the differential equation $\frac{dT}{dt} = k(T - 22)$ where k is a constant.

a Solve this equation to show that $T = Ae^{kt} + 22$ where A is a constant.

b When $t = 0$, $T = 100$ and when $t = 15$, $T = 70$.

i Use this information to find the value of A and of k.

ii Hence find the value of t when $T = 40$. [IB May 04 P1 Q4]

8 The velocity of a particle is given by the formula $v = \frac{1}{t^2\sqrt{t - 1}}$ for $t > 1$. Using the substitution $t = \sec^2\theta$, find the displacement travelled between $t = 2$ seconds and $t = T$ seconds.

9 Consider the curve $y = -\frac{r}{h}x + r$. The triangular region of this curve which occupies the first quadrant is rotated fully about the x-axis. Show that the volume of the cone formed is $\frac{1}{3}\pi r^2 h$.

10 A sample of radioactive material decays at a rate which is proportional to the amount of material present in the sample. Find the half-life of the material if 50 grams decay to 48 grams in 10 years. [IB Nov 01 P1 Q19]

11 The acceleration in ms^{-2} of a particle moving in a straight line at time t, is given by the formula $a = \sin\frac{2\pi}{3}t$. The particle starts from rest from a point where its displacement is 0.5 m from a fixed point O on the line.

a Find the velocity and displacement of the particle from O at any time t.

b Find the time that elapses before the particle comes to rest again.

12 **a** Let $y = \sin(kx) - kx\cos(kx)$, where k is a constant.

Show that $\frac{dy}{dx} = k^2 x\sin(kx)$.

A particle is moving along a straight line so that t seconds after passing through a fixed point O on the line, its velocity $v(t)$ ms^{-1} is given by $v(t) = t\sin\left(\frac{\pi}{3}t\right)$.

b Find the values of t for which $v(t) = 0$, given that $0 \le t \le 6$.

c **i** Write down a mathematical expression for the **total** distance travelled by the particle in the first six seconds after passing through O.

ii Find this distance. [IB Nov 01 P2 Q2]

13 When air is released from an inflated balloon it is found that the rate of decrease of the volume of the balloon is proportional to the volume of the balloon. This can be represented by the differential equation $\frac{dv}{dt} = -kv$, where v is the volume, t is the time and k is the constant of proportionality.

a If the initial volume of the balloon is v_0, find an expression, in terms of k, for the volume of the balloon at time t.

b Find an expression, in terms of k, for the time when the volume is $\frac{v_0}{2}$.

[IB May 99 P1 Q19]

14 Show by means of the substitution $x = \tan\theta$ that

$\int_0^1 \frac{1}{(x^2+1)^2}\,dx = \int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$. Hence find the exact value of the volume formed when the curve $y = \frac{1}{x^2+1}$ bounded by the lines $x = 0$ and $x = 1$ is rotated fully about the x-axis.

15 Consider the curve $y^2 = 9a(4a - x)$.

a Sketch the part of the curve that lies in the first quadrant.

b Find the exact value of the volume V_x when this part of the curve is rotated through 360° about the x-axis.

c Show that $\frac{V_y}{V_x} = \frac{a}{b}$ where V_y is the volume generated when the curve is rotated fully about the y-axis and a and b are integers.