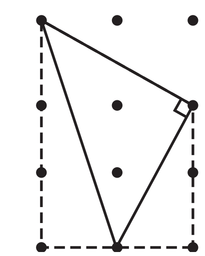
7 Trigonometry 2

Consider the equation below. If all angles are in radians

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$

Can you prove this?



The mathematics behind the fact

The hint is in the diagram, since we can check that the angles of the three triangles at their common vertex add up to π. Is it possible to find a similar proof for the following equation?

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

http://www.math.hmc.edu/funfacts/ffiles/20005.2.shtml

Accessed 1 Dec 06

In Chapter 1, we met the trigonometric functions and their graphs. In this chapter we will meet some trigonometric identities. These can be used to solve problems and are also used to prove other trigonometric results.

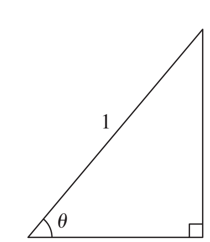
7.1 Identities

An identity is a result or equality that holds true regardless of the value of any of the variables within it.

$\equiv$ means "is identical to" although the equals sign is often still used in identity work.

We met an example of a trigonometric identity in Chapter 1: $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$

Pythagorean identities

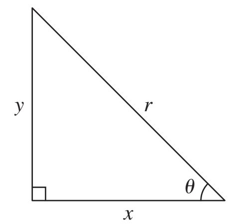


Applying Pythagoras' theorem to the right-angled triangle obtained from the unit circle,

$$y^2 + x^2 = 1 \Rightarrow \sin^2\theta + \cos^2\theta = 1$$

This is an identity and so strictly should be written $\sin^2\theta + \cos^2\theta \equiv 1$.

This resulted from the definition of $\sin\theta$ and $\cos\theta$ from the unit circle. We can also see that it is true from a right-angled triangle.



We know that $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$.

By Pythagoras' theorem, $x^2 + y^2 = r^2$.

$$\begin{aligned}\text{So } \sin^2\theta + \cos^2\theta &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2\\ &= \frac{y^2 + x^2}{r^2}\\ &= \frac{r^2}{r^2}\\ &= 1\end{aligned}$$

Finding $\cos\theta$ given $\sin\theta$

We now know that $\sin^2\theta + \cos^2\theta = 1$.

This is often expressed as $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$.

Example

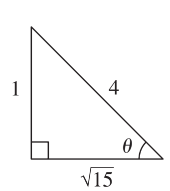
If $\sin\theta = \dfrac{1}{4}$, find possible values of $\cos\theta$.

Since $\sin\theta = \dfrac{1}{4}$, $\sin^2\theta = \dfrac{1}{16}$

$$\begin{aligned}\Rightarrow \cos^2\theta &= 1 - \frac{1}{16}\\ &= \frac{15}{16}\\ \Rightarrow \cos\theta &= \pm\frac{\sqrt{15}}{4}\end{aligned}$$

As we do not know which quadrant θ lies in, $\cos\theta$ could be positive or negative.

The information could also be displayed in a triangle, and $\cos\theta$ calculated that way.



Example

If $\cos\theta = -\dfrac{1}{2}$, find possible values of $\sin\theta$.

$$\begin{aligned}\cos^2\theta &= \frac{1}{4}\\ \Rightarrow \sin^2\theta &= 1 - \frac{1}{4}\\ &= \frac{3}{4}\\ \Rightarrow \sin\theta &= \pm\frac{\sqrt{3}}{2}\end{aligned}$$

Taking this identity of $\sin^2\theta + \cos^2\theta = 1$, we can create other identities that are useful in trigonometric work.

By dividing both sides by $\cos^2\theta$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

So

$$\tan^2\theta + 1 \equiv \sec^2\theta$$

Similarly, by dividing both sides by $\sin^2\theta$ we obtain

$$1 + \cot^2\theta \equiv \csc^2\theta$$

Example

Simplify $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta}$.

$$\begin{aligned}\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} &= \frac{\sin^2\theta}{\sin\theta(1 + \cos\theta)} + \frac{(1 + \cos\theta)^2}{\sin\theta(1 + \cos\theta)}\\ &= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1 + \cos\theta)}\\ &= \frac{2 + 2\cos\theta}{\sin\theta(1 + \cos\theta)}\\ &= \frac{2}{\sin\theta}\end{aligned}$$

Identities are often used to simplify expressions.

Questions involving simplification can also be presented as proving another identity.

Example

Prove that $\dfrac{\sin^4\theta}{\tan\theta} \equiv \sin\theta\cos\theta - \sin\theta\cos^3\theta$.

$$\begin{aligned}
\text{LHS} &\equiv \frac{\sin^4\theta}{\tan\theta}\\
&\equiv \frac{\sin^4\theta\cos\theta}{\sin\theta}\\
&\equiv \sin^3\theta\cos\theta\\
&\equiv \sin\theta\sin^2\theta\cos\theta\\
&\equiv \sin\theta\cos\theta(1-\cos^2\theta)\\
&\equiv \sin\theta\cos\theta - \sin\theta\cos^3\theta\\
&\equiv \text{RHS}
\end{aligned}$$

When faced with this type of question, the strategy is to take one side (normally the left-hand side) and work it through to produce the right-hand side.

Example

Solve $3\sec^2\theta + 2\tan\theta - 4 = 0$ for $0 \le \theta < 2\pi$.

Using $\sec^2\theta \equiv \tan^2\theta + 1$ this becomes

$3(\tan^2\theta + 1) + 2\tan\theta - 4 = 0$

$\Rightarrow 3\tan^2\theta + 2\tan\theta - 1 = 0$

By using the identity, the equation has been transformed into one that involves only one type of trigonometric function, $\tan\theta$.

This is a quadratic equation where the variable is $\tan\theta$. It can be solved using factorisation or the quadratic formula.

Here, $3\tan^2\theta + 2\tan\theta - 1 = 0$

$\Rightarrow (3\tan\theta - 1)(\tan\theta + 1) = 0$

$\Rightarrow \tan\theta = \dfrac{1}{3}$ or $\tan\theta = -1$

$\Rightarrow \theta = 0.322, 1.01$ or $\theta = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

Example

Solve $\tan\theta + 3\cot\theta = 5\sec\theta$ for $0 \le \theta < 2\pi$.

Here we can use the definitions of each of these three trigonometric functions to simplify the equation.

$$\frac{\sin\theta}{\cos\theta} + \frac{3\cos\theta}{\sin\theta} = \frac{5}{\cos\theta}$$

$$\Rightarrow \frac{\sin\theta - 5}{\cos\theta} = \frac{-3\cos\theta}{\sin\theta}$$

$\Rightarrow \sin^2\theta - 5\sin\theta = -3\cos^2\theta$

$\Rightarrow \sin^2\theta - 5\sin\theta = -3(1 - \sin^2\theta)$

$\Rightarrow 2\sin^2\theta + 5\sin\theta - 3 = 0$

$\Rightarrow (2\sin\theta - 1)(\sin\theta + 3) = 0$

$\Rightarrow \sin\theta = \dfrac{1}{2}$ or $\sin\theta = -3$

Since $\sin\theta = -3$ has no solution, the solution to the equation is $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

Exercise 1

These Pythagorean identities can also be used to help solve trigonometric equations.

1 For the given values of $\sin\theta$, give possible values of $\cos\theta$.

a $\sin\theta = \dfrac{1}{2}$ **b** $\sin\theta = \dfrac{1}{\sqrt{2}}$ **c** $\sin\theta = \dfrac{5}{7}$ **d** $\sin\theta = 3$

2 For the given values of $\cos\theta$, give possible values of $\sin\theta$.

a $\cos\theta = \dfrac{\sqrt{3}}{2}$ **b** $\cos\theta = \dfrac{4}{5}$ **c** $\cos\theta = 4$ **d** $\cos\theta = 0.2$

3 Prove the following to be true, using trigonometric identities.

a $\cos^3\theta\tan\theta \equiv \sin\theta - \sin^3\theta$

b $\cos^5\theta \equiv \cos\theta - 2\sin^2\theta\cos\theta$

c $(4\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 4\cos\theta)^2 \equiv 25$

d $(\sec\theta - 1)^2 - 2\sec\theta \equiv \tan^2\theta$

4 If $\cot^2\theta = 8$, what are possible values for $\sin\theta$?

5 Simplify these.

a $\dfrac{6 - 6\cos^2\theta}{2\sin\theta}$ **b** $\dfrac{\sin^3\theta + \sin\theta\cos^2\theta}{\cos\theta}$

c $\dfrac{\tan^2\theta - \sec^2\theta}{\csc\theta}$ **d** $\dfrac{7\cos^4\theta + 7\sin^2\theta\cos^2\theta}{\sin^2\theta}$

6 Prove that $\tan^2\phi + \cot^2\phi \equiv \sec^2\phi + \csc^2\phi - 2$.

7 Solve these following equations for $0 \le \theta < 2\pi$.

a $\csc^2\theta + \cot^2\theta = 5$ **b** $\sin^2\theta + 2\cos\theta - 1 = 0$

c $3\tan\theta = 4\sec^2\theta - 5$ **d** $\cot\theta + \tan\theta = 2$

e $6\cot^2\theta + 13\csc\theta - 2 = 0$ **f** $\cos\theta - 2\sin^2\theta = -1$

7.2 Compound angle (addition) formulae

These formulae allow the expansion of expressions such as $\sin(A + B)$.

It is very important to recognize that $\sin(A + B) \neq \sin A + \sin B$.

This becomes clear by taking a simple example:

$\sin 90° = \sin(30 + 60)° = 1$

But $\sin 30° + \sin 60° = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} \simeq 1.37\ (\neq 1)$

The six addition formulae are:

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$$
$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$
$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

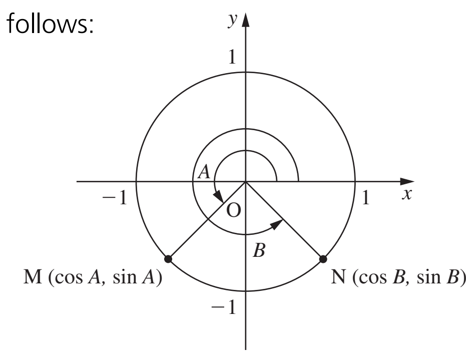
The formula for $\cos(A-B)$ can be proved as follows:

Take two points on the unit circle, as shown.

These two points have coordinates

M($\cos A$, $\sin A$) and N($\cos B$, $\sin B$).

The square of the distance from M to N is given by Pythagoras' theorem:



$$MN^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B$$
$$= 1 + 1 - 2\cos A \cos B - 2 \sin A \sin B$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B)$$

Using the cosine rule in triangle MON:

$$MN^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(A-B)$$
$$= 2 - 2\cos(A-B)$$

Angle MON = angle A − angle B

Equating these two results for MN^2:

$$2 - 2\cos(A-B) = 2 - 2(\cos A \cos B + \sin A \sin B)$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

The other formulae are easily proved from this one. The starting point for each is given below:

$$\cos(A+B) = \cos[A - (-B)]$$

$$\sin(A-B) = \cos\left[\frac{\pi}{2} - (A-B)\right] \qquad \sin(A+B) = \sin[A-(-B)]$$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} \qquad \tan(A+B) = \tan[A-(-B)]$$

Example

Using $15° = 60° - 45°$, find the exact value of $\cos 15°$.

$$\cos 15° = \cos(60 - 45)°$$
$$= \cos 60° \cos 45° + \sin 60° \sin 45°$$
$$= \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

This is a typical non-calculator question.

Example

Prove that $2\cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta - \sqrt{3}\sin\theta$.

$$\text{LHS} = 2\cos\left(\theta + \frac{\pi}{3}\right)$$
$$= 2\left(\cos\theta \cos\frac{\pi}{3} - \sin\theta \sin\frac{\pi}{3}\right)$$
$$= 2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)$$
$$= \cos\theta - \sqrt{3}\sin\theta = \text{RHS}$$

Example

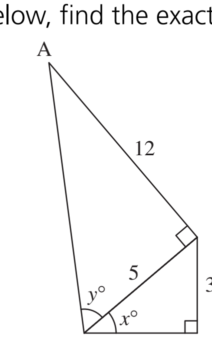
Simplify $\dfrac{\tan 96° - \tan 51°}{1 + \tan 96° \tan 51°}$.

Recognizing this is of the form $\tan(A-B)$, it can be written

$$\frac{\tan 96° - \tan 51°}{1 + \tan 96° \tan 51°} = \tan(96 - 51)°$$
$$= \tan 45°$$
$$= 1$$

Example

In the diagram below, find the exact value of $\cos A\hat{B}C$.



In the diagram, $A\hat{B}C = x° + y°$.

Using Pythagoras' theorem, we can calculate BC to be 4 and AB to be 13.

Hence $\cos x° = \frac{4}{5}$, $\cos y° = \frac{5}{13}$

$\sin x° = \frac{3}{5}$, $\sin y° = \frac{12}{13}$

$$
\begin{aligned}
\text{So}\quad \cos A\hat{B}C &= \cos(x + y)° \\
&= \cos x° \cos y° - \sin x° \sin y° \\
&= \left(\frac{4}{5} \times \frac{5}{13}\right) - \left(\frac{3}{5} \times \frac{12}{13}\right) \\
&= \frac{20}{65} - \frac{36}{65} \\
&= -\frac{16}{65}
\end{aligned}
$$

For questions of this type it is worth remembering the Pythagorean triples (and multiples thereof) such as:

3, 4, 5
5, 12, 13
8, 15, 17
7, 24, 25

These identities can also be employed to solve equations, as in the following examples.

Example

Solve $\cos\theta = \sin\left(\theta + \frac{\pi}{3}\right)$ for $0 \leq \theta < 2\pi$.

$$
\begin{aligned}
\cos\theta &= \sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3} \\
\Rightarrow \cos\theta &= \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta \\
\Rightarrow \frac{2 - \sqrt{3}}{2}\cos\theta &= \frac{1}{2}\sin\theta \\
\Rightarrow \tan\theta &= 2 - \sqrt{3} \\
\Rightarrow \theta &= 0.262,\ 3.40
\end{aligned}
$$

The difference between this type of question and an identity should be noted. An identity holds true for all values of θ, whereas this equation is true only for certain values of θ.

Example

Solve $\cos(45 - x)° = \sin(30 + x)°$ for $0° \leq x° < 360°$.

$$
\begin{aligned}
\cos 45° \cos x° + \sin 45° \sin x° &= \sin 30° \cos x° + \cos 30° \sin x° \\
\Rightarrow \frac{1}{\sqrt{2}}\cos x° + \frac{1}{\sqrt{2}}\sin x° &= \frac{1}{2}\cos x° + \frac{\sqrt{3}}{2}\sin x° \\
\Rightarrow \frac{\sqrt{2} - 1}{2}\cos x° &= \frac{\sqrt{3} - \sqrt{2}}{2}\sin x° \\
\Rightarrow \tan x° &= \frac{\sqrt{2} - 1}{\sqrt{3} - \sqrt{2}} \\
\Rightarrow x° &= 52.5°,\ 232.5°
\end{aligned}
$$

Exercise 2

1 By considering $75° = 30° + 45°$, find these.
a $\sin 75°$ **b** $\cos 75°$ **c** $\tan 75°$

2 Find $\sin 15°$ by calculating
a $\sin(60 - 45)°$ **b** $\sin(45 - 30)°$

3 Find the exact value of $\cos 105°$.

4 Find $\cos\frac{11\pi}{12}$ by using the fact that $\frac{11\pi}{12} = \frac{\pi}{4} + \frac{2\pi}{3}$.

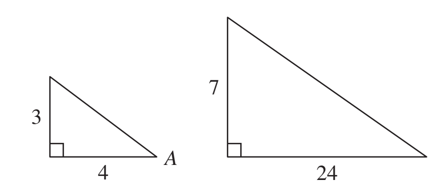
5 Prove that $4\sin\left(\theta - \frac{\pi}{6}\right) \equiv 2\sqrt{3}\sin\theta - 2\cos\theta$.

6 Prove that $\sin(x + y)\sin(x - y) \equiv \sin^2 x - \sin^2 y$.

7 Prove that $\csc\left(\frac{\pi}{2} + \theta\right) \equiv \sec\theta$.

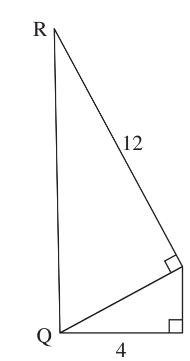
8 Simplify these.
a $\cos 310° \cos 40° + \sin 310° \sin 40°$
b $\sin\frac{\pi}{2}\cos\frac{11\pi}{6} + \sin\frac{11\pi}{6}\cos\frac{\pi}{2}$

9 A and B are acute angles as shown.

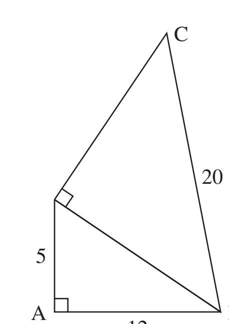


Find these.
a $\sin(A + B)$ **b** $\cos(A - B)$ **c** $\tan(B - A)$

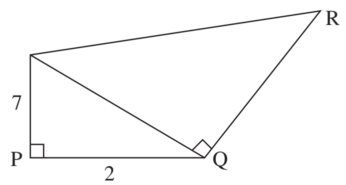
10 In the diagram below, find $\cos P\hat{Q}R$.



11 In the diagram below, find $\sin A\hat{B}C$.



12 In the diagram below, find $\tan P\hat{Q}R$.



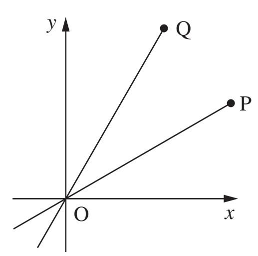
13 A is acute, B is obtuse, $\sin A = \frac{3}{7}$ and $\sin B = \frac{2}{3}$. Without using a calculator, find the possible values of $\sin(A + B)$ and $\cos(A + B)$.

14 Solve $\cos(45 - x)° = \sin(30 + x)°$ for $0° \le x° < 360°$.

15 Show that $\cos\left(\frac{\pi}{4} + \theta\right) - \sin\left(\frac{\pi}{4} - \theta\right) = 0$.

16 The gradients of OP and OQ are $\frac{1}{3}$ and 3 respectively.

Find $\cos P\hat{O}Q$.



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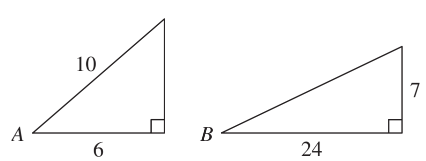


For the triangles above, and by considering $2A = A + A$, find these.

a $\sin 2A$ **b** $\cos 2A$ **c** $\sin 2B$ **d** $\cos 2B$

18 Solve the following equations for $0° \le x° < 360°$.

a $\sin(x + 30)° = 2\cos x°$

b $\cos(x + 45)° = \sin(x + 45)°$

c $6\sin x° = \cos(x + 30)°$

19 Solve the following equations for $0 \le \theta < 2\pi$.

a $\sin\theta + \cos\theta = \cos\left(\theta - \frac{\pi}{6}\right)$

b $\tan\left(\theta + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}$

c $\cos\left(x + \frac{2\pi}{3}\right) = \sin\left(x + \frac{3\pi}{4}\right)$

7.3 Double angle formulae

It is useful to consider the special cases of addition formulae that are the double angle formulae.

$$\begin{aligned}\sin 2A &= \sin(A + A)\\ &= \sin A\cos A + \cos A\sin A\\ &= 2\sin A\cos A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos(A + A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

$$\begin{aligned}\tan 2A &= \tan(A + A)\\ &= \frac{\tan A + \tan A}{1 - \tan A\tan A}\\ &= \frac{2\tan A}{1 - \tan^2 A}\end{aligned}$$

It is often useful to rearrange $\cos 2A$ by using the identity $\sin^2 A + \cos^2 A = 1$.

$$\begin{aligned}\text{So }\cos 2A &= \cos^2 A - \sin^2 A\\ &= 1 - \sin^2 A - \sin^2 A\\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\begin{aligned}\text{Or }\cos 2A &= \cos^2 A - \sin^2 A\\ &= \cos^2 A - (1 - \cos^2 A)\\ &= 2\cos^2 A - 1\end{aligned}$$

In this chapter we have met the Pythagorean identities, compound angle formulae and double angle formulae. These are all summarized below.

Pythagorean identities

$\sin^2\theta + \cos^2\theta \equiv 1$

$\tan^2\theta + 1 \equiv \sec^2\theta$

$1 + \cot^2\theta \equiv \csc^2\theta$

Compound angle formulae

$\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$

$\sin(A - B) \equiv \sin A\cos B - \cos A\sin B$

$\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$

$\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$

$\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Double angle formulae

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ \cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\\ \tan 2A &= \frac{2\tan A}{1 - \tan^2 A}\end{aligned}$$

Example

By considering $\frac{\pi}{2}$ as $2\left(\frac{\pi}{4}\right)$, use the double angle formulae to find these.

(a) $\sin\frac{\pi}{2}$ (b) $\cos\frac{\pi}{2}$ (c) $\tan\frac{\pi}{2}$

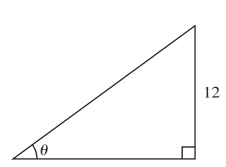
(a) $$\begin{aligned}\sin\frac{\pi}{2} &= 2\sin\frac{\pi}{4}\cos\frac{\pi}{4}\\ &= 2\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\\ &= 2\times\frac{1}{2}\\ &= 1\end{aligned}$$

(b) $$\begin{aligned}\cos\frac{\pi}{2} &= \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4}\\ &= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2\\ &= 0\end{aligned}$$

(c) $$\begin{aligned}\tan\frac{\pi}{2} &= \frac{2\tan\frac{\pi}{4}}{1-\tan^2\frac{\pi}{4}}\\ &= \frac{2}{1-1}\\ &= \frac{2}{0}\\ &= \infty\end{aligned}$$

Example

Find $\cos 2\theta$.



Clearly $\cos\theta = \frac{9}{15}$

So $$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{9}{15}\right)^2 - 1\\ &= \frac{162}{225} - 1\\ &= -\frac{63}{225}\end{aligned}$$

Example

Find an expression for $\sin 3\theta$ in terms of $\sin\theta$.

$$\begin{aligned}\sin 3\theta &= \sin(2\theta+\theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &= 2\sin\theta\cos\theta\cos\theta + (1-2\sin^2\theta)\sin\theta\\ &= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta\\ &= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta\\ &= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Example

Find an expression for $\cos 4\theta$ in terms of $\cos\theta$.

Here we can use the double angle formula – remember that the double angle formulae do not apply only to 2θ; they work for any angle that is twice the size of another angle.

$$\begin{aligned}\cos 4\theta &= 2\cos^2 2\theta - 1\\ &= 2(2\cos^2\theta - 1)^2 - 1\\ &= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

Example

Prove the identity $\frac{1-\tan^2\theta}{1+\tan^2\theta} \equiv \cos 2\theta$.

$$\begin{aligned}\text{LHS} &= \frac{1-\tan^2\theta}{1+\tan^2\theta}\\ &\equiv \frac{1-\tan^2\theta}{\sec^2\theta}\\ &\equiv \cos^2\theta\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)\\ &\equiv \cos^2\theta - \sin^2\theta\\ &\equiv \cos 2\theta = \text{RHS}\end{aligned}$$

Example

Prove the identity $\tan 3\alpha + \tan\alpha \equiv \frac{\sin 4\alpha}{\cos 3\alpha\cos\alpha}$.

In this example, it is probably easiest to begin with the right-hand side and show that it is identical to the left-hand side. Although it involves what appears to be a double angle (4α), in fact it is not useful to use the double angle formulae.

To recognize this, it is important to look at the other side of the identity and realize what the goal is.

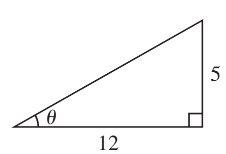
$$\begin{aligned}\text{RHS} &= \frac{\sin 4\alpha}{\cos 3\alpha \cos \alpha}\\ &\equiv \frac{\sin(3\alpha + \alpha)}{\cos 3\alpha \cos \alpha}\\ &= \frac{\sin 3\alpha \cos \alpha + \cos 3\alpha \sin \alpha}{\cos 3\alpha \cos \alpha}\\ &= \frac{\sin 3\alpha \cos \alpha}{\cos 3\alpha \cos \alpha} + \frac{\cos 3\alpha \sin \alpha}{\cos 3\alpha \cos \alpha}\\ &= \tan 3\alpha + \tan \alpha = \text{LHS}\end{aligned}$$

Exercise 3

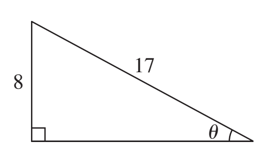
1 Use the double angle formulae to find these.

a $\sin\frac{2\pi}{3}$ **b** $\cos\frac{5\pi}{3}$ **c** $\tan\frac{2\pi}{3}$

2 Find $\sin 2\theta$.



3 Find $\cos 2\theta$.



4 Given that θ is an acute angle with $\tan\theta = \frac{1}{2}$, calculate the exact value of these.

a $\sin\theta$ **b** $\cos\theta$

5 Find an expression for $\cos 5\theta$ in terms of $\sin\theta$ and $\cos\theta$.

6 Find an expression for $\sin 6\theta$ in terms of $\sin\theta$.

7 Find the exact value of $\cos 2\theta$ given that $\sin\theta = \frac{12}{13}$ (θ is not acute).

8 Prove that $\tan\theta \equiv \dfrac{1 - \cos 2\theta}{\sin 2\theta}$.

9 Prove that $\dfrac{1 + \cos 2y}{\sin 2y} \equiv \dfrac{\sin 2y}{1 - \cos 2y}$.

10 Prove that $\cos\phi + \sin\phi \equiv \dfrac{\cos 2\phi}{\cos\phi - \sin\phi}$.

11 Show that $\cos 3\theta - \sin 4\theta = 4\cos^3\theta(1 - 2\sin\theta) + \cos\theta(4\sin\theta - 3)$.

12 Prove that $\dfrac{\cos A + \sin A}{\cos A - \sin A} \equiv \sec 2A + \tan 2A$.

13 Prove that $\sin 2\alpha \equiv \dfrac{2\tan\alpha}{1 + \tan^2\alpha}$.

7.4 Using double angle formulae

Half angle formulae

It is useful to rearrange the double angle formulae to obtain formulae for a half angle. We have seen that double angle formulae can be applied to different angles.

So we can find expressions for $\cos\frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta$.

These formulae are particularly applied when integrating trigonometric functions (see Chapter 15).

$\cos 2\theta = 2\cos^2\theta - 1$ can be rearranged to

$$\begin{aligned}&2\cos^2\theta = \cos 2\theta + 1\\ \Rightarrow\ &\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)\\ \Rightarrow\ &\cos\theta = \pm\sqrt{\frac{1}{2}(\cos 2\theta + 1)}\end{aligned}$$

So $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1}{2}(\cos\theta + 1)}$

Similarly

$$\begin{aligned}&1 - 2\sin^2\theta = \cos 2\theta\\ \Rightarrow\ &2\sin^2\theta = 1 - \cos 2\theta\\ \Rightarrow\ &\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)\\ \Rightarrow\ &\sin\theta = \pm\sqrt{\frac{1}{2}(1 - \cos 2\theta)}\end{aligned}$$

So $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1}{2}(1 - \cos\theta)}$

Example

Find the exact value of $\cos 15°$.

$$\begin{aligned}\cos 15° &= \sqrt{\frac{1}{2}(\cos 30° + 1)}\\ &= \sqrt{\frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right)}\\ &= \sqrt{\frac{1}{2}\left(\frac{2 + \sqrt{3}}{2}\right)}\\ &= \sqrt{\frac{2 + \sqrt{3}}{4}}\\ &= \frac{\sqrt{2 + \sqrt{3}}}{2}\end{aligned}$$

Trigonometric equations involving double angles

We covered basic equations involving double angles that can be solved without using the double angle formulae in Chapter 1.

Example

Solve $\sin 2x^\circ = \dfrac{\sqrt{3}}{2}$ for $0^\circ \le x^\circ < 360^\circ$.

✓ S	✓ A
T	C

$\sin 2x^\circ = \dfrac{\sqrt{3}}{2}$

$\Rightarrow 2x^\circ = 60^\circ, 120^\circ$

$\Rightarrow x^\circ = 30^\circ, 60^\circ, 210^\circ, 240^\circ$

However, if there is another trigonometric term involved (and we cannot use a calculator to solve the equation), then factorisation methods need to be employed.

Example

Solve $\sin 2\theta - \sin\theta = 0$ for $0 \le \theta < 2\pi$.

$$\begin{aligned} \sin 2\theta - \sin\theta &= 0 \\ \Rightarrow 2\sin\theta\cos\theta - \sin\theta &= 0 \\ \Rightarrow \sin\theta(2\cos\theta - 1) &= 0 \\ \Rightarrow \sin\theta = 0 \text{ or } 2\cos\theta - 1 &= 0 \end{aligned}$$

$\Rightarrow \theta = 0, \pi \qquad \Rightarrow \cos\theta = \dfrac{1}{2}$

$\Rightarrow \theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

So $\theta = 0, \dfrac{\pi}{3}, \pi$ or $\dfrac{5\pi}{3}$.

Example

Solve $\cos 2x^\circ - 5\cos x^\circ = 2$ for $0^\circ \le x^\circ < 360^\circ$.

$$\begin{aligned} \cos 2x^\circ - 5\cos x^\circ &= 2 \\ \Rightarrow 2\cos^2 x^\circ - 1 - 5\cos x^\circ &= 2 \\ \Rightarrow 2\cos^2 x^\circ - 5\cos x^\circ - 3 &= 0 \\ \Rightarrow (2\cos x^\circ + 1)(\cos x^\circ - 3) &= 0 \\ \Rightarrow 2\cos x^\circ + 1 = 0 \text{ or } \cos x^\circ - 3 &= 0 \end{aligned}$$

$\Rightarrow \cos x^\circ = -\dfrac{1}{2}$ or $\cos x^\circ = 3$

$\Rightarrow x^\circ = 120^\circ, 240^\circ$

$\cos x^\circ = 3$ has no solution.

Example

Solve $3\cos 2\theta + 11\sin\theta = -4$ for $0 \le \theta < 2\pi$.

$$\begin{aligned} &\Rightarrow 3(1 - 2\sin^2\theta) + 11\sin\theta + 4 = 0 \\ &\Rightarrow 3 - 6\sin^2\theta + 11\sin\theta + 4 = 0 \\ &\Rightarrow 6\sin^2\theta - 11\sin\theta - 7 = 0 \\ &\Rightarrow (3\sin\theta - 7)(2\sin\theta + 1) = 0 \end{aligned}$$

$\Rightarrow \sin\theta = \dfrac{7}{3}$ or $\sin\theta = -\dfrac{1}{2}$

$\Rightarrow \theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

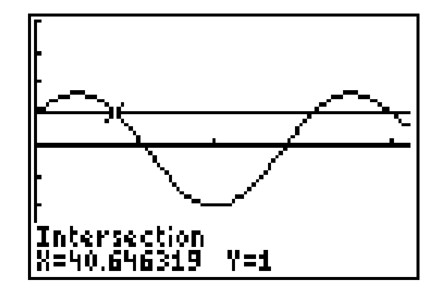
The form of $\cos 2\theta$ required is determined by the other term in the equation (sin or cos).

$\sin\theta = \dfrac{7}{3}$ has no solution.

An equation of this type is often solved using a calculator (when it is available). In fact, in some cases this is the only appropriate method.

Example

Solve $\sin 3x^\circ + \cos 2x^\circ = 1$ for $0^\circ \le x^\circ < 180^\circ$.



$x^\circ = 0^\circ, 40.6^\circ, 139^\circ$

Exercise 4

1 Prove $\tan\left(\dfrac{\theta}{2}\right) \equiv \csc\theta - \cot\theta$.

2 Prove $2\sin^2\dfrac{\theta}{2} \equiv 1 - \cos\theta$.

3 Find the value of sin 75°, using a half angle formula.

4 Find the exact value of $\cos\dfrac{\pi}{8}$, using a half angle formula.

5 Solve these equations for $0 \le \theta < 2\pi$.

a $\sin 2\theta = \dfrac{\sqrt{3}}{2}$ **b** $6\cos 2\theta + 1 = 4$

6 Solve these equations for $0^\circ \le x^\circ < 360^\circ$.

a $\sin 2x^\circ - \cos x^\circ = 0$ **b** $\sin 2x^\circ - 4\sin x^\circ = 0$

c $\cos 2x^\circ - \cos x^\circ + 1 = 0$ **d** $\cos 2x^\circ - 4\sin x^\circ + 5 = 0$

e $\cos 2x^\circ + 5\cos x^\circ - 2 = 0$ **f** $\cos 2x^\circ + 3\cos x^\circ - 1 = 0$

g $2\cos 2x^\circ + \cos x^\circ - 1 = 0$ **h** $\cos 2 = 7\sin x^\circ + 4$

7 Solve these equations for $0 \le \theta < 2\pi$.

a $\cos 2\theta + \cos\theta = 0$ **b** $\sin 2\theta + \sin\theta = 0$

c $2\cos 2\theta + 1 = 0$ **d** $\sin 2\theta - \cos\theta = 0$

e $\cos 2\theta = \cos\theta$ **f** $\cos 2\theta = 4\cos\theta + 5$

g $\cos 2\theta = \sin\theta + 1$

8 Solve these equations for $-\pi \le \theta < \pi$.

a $\sin 2\theta = 2\sin\theta$ **b** $\cos 2\theta = 1 - 3\cos\theta$

c $2\cos\theta + 2 = \cos 2\theta + 4$ **d** $\cos 2\theta + 3\cos\theta + 2 = 0$

9 Solve these equations for $0 \le \theta < 2\pi$. Give your answers to 3 sf.

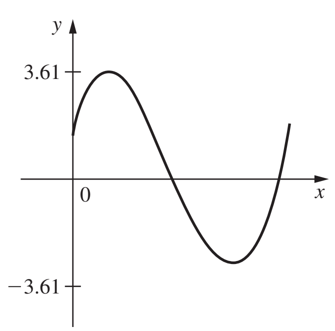
a $6\cos 2\theta - 5\cos\theta + 4 = 0$ **b** $2\cos 2\theta - 3\sin\theta + 1 = 0$

7.5 Wave function

This refers to functions of the type $f(x) = a\cos x° + b\sin x°$, where there are both sine and cosine terms in one function.

What does the graph of this type of function look like?

This is the graph of $y = 2\cos x° + 3\sin x°$:



Although this part of trigonometry is not explicitly stated as part of the IB HL syllabus, it is really an application of compound angle formulae and is worth studying.

It appears to have all of the properties of a single trigonometric function, in that it is a periodic wave with symmetrical features. As its maximum and minimum values are numerically the same, we can conclude that there is a stretch factor involved. As the graph begins neither on the x-axis nor at a maximum/minimum, there must be a horizontal shift.

This suggests a function of the form $k\cos(x - \alpha)°$.

We can check this by finding k and $\alpha°$.

Example

Express $f(x) = 2\cos x° + 3\sin x°$ in the form $k\cos(x - \alpha)°$.

We know that $k\cos(x - \alpha)° = k\cos x°\cos\alpha° + k\sin x°\sin\alpha°$, so for $f(x)$ to be expressed in this form

$2\cos x° + 3\sin x° = k\cos x°\cos\alpha° + k\sin x°\sin\alpha°$

Comparing the two sides, we can conclude that

$k\sin\alpha° = 3$

$k\cos\alpha° = 2$

By squaring and adding, we can find k:

$$k^2\sin^2\alpha° + k^2\cos^2\alpha° = 3^2 + 2^2$$
$$\Rightarrow k^2(\sin^2\alpha° + \cos^2\alpha°) = 13$$
$$\Rightarrow k^2 = 13$$
$$\Rightarrow k = \sqrt{13}$$

By dividing, we can find $\alpha°$:

$$\frac{k\sin\alpha°}{k\cos\alpha°} = \tan\alpha° = \frac{3}{2}$$
$$\Rightarrow \alpha° = \tan^{-1}\frac{3}{2}$$
$$\Rightarrow \alpha° = 56.3°$$

It is vital to consider which quadrant $\alpha°$ lies in. This is best achieved using the diagram.

S	A
T	C

In this case, we know that

$k\sin\alpha° = +$

$k\cos\alpha° = +$

S ✓	A ✓✓
T	C ✓

Since $k > 0$ (always), we can see that the quadrant with two ticks in it is the first quadrant.

So here $\alpha°$ is acute.

Hence $f(x) = 2\cos x° + 3\sin x° = \sqrt{13}\cos(x - 56.3)°$.

Method for wave function

To express a function of the form $f(\theta) = a\sin\theta + b\cos\theta$ as a single trigonometric function:

1. Expand the desired form using the compound formula (if no form is given, use $k\cos(x - \alpha)°$).
2. Compare the two sides to find $k\sin\alpha°$ and $k\cos\alpha°$ (write them in this order).
3. Square and add to find k^2.
4. Divide to find $\tan\alpha°$.
5. Use the positivity diagram to find $\alpha°$.

Example

Express $3\cos x° - 4\sin x°$ as $k\cos(x - \alpha)°, k > 0, 0° \le \alpha° < 360°$.

Let $3\cos x° - 4\sin x° = k\cos(x - \alpha)°$
$= k\cos x°\cos\alpha° + k\sin x°\sin\alpha°$

So $k\sin\alpha° = -4$

$k\cos\alpha° = 3$

Squaring and adding:

$$k^2\sin^2\alpha° + k^2\cos^2\alpha° = (-4)^2 + 3^2$$
$$k^2 = 25$$
$$k = 5$$

Dividing:

$$\frac{k\sin\alpha°}{k\cos\alpha°} = \tan\alpha° = -\frac{4}{3}$$

$$\tan^{-1}\left(\frac{4}{3}\right) = 53.1°$$

S	A ✓
T ✓	C ✓✓

So $\alpha° = 360° - 53.1°$
$= 306.9°$

Hence $3\cos x° - 4\sin x° = 5\cos(x - 306.9)°$

It is not always the form $k\cos(x-\alpha)°$ that is required, but the method is precisely the same.

Example

Express $\cos\theta + \sqrt{3}\sin\theta$ as $k\sin(\theta-\alpha)$, $k > 0$, $0 \leq \alpha < 2\pi$.

Let $\cos\theta + \sqrt{3}\sin\theta = k\sin(\theta-\alpha)$
$$= k\sin\theta\cos\alpha - k\cos\theta\sin\alpha$$

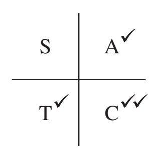
So $-k\sin\alpha = 1$
$\Rightarrow k\sin\alpha = -1$
and $k\cos\alpha = \sqrt{3}$

Squaring and adding:

$$k^2\sin^2\alpha + k^2\cos^2\alpha = (-1)^2 + \sqrt{3}^2$$
$$k^2 = 4$$
$$k = 2$$

Dividing:

$$\frac{k\sin\alpha}{k\cos\alpha} = \tan\alpha = -\frac{1}{\sqrt{3}}$$
$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$



So $\alpha = 2\pi - \dfrac{\pi}{6}$
$= \dfrac{11\pi}{6}$

Hence $\cos\theta + \sqrt{3}\sin\theta = 2\sin\left(\theta - \dfrac{11\pi}{6}\right)$

This method works in the same way for functions involving a multiple angle (as long as both the sine and cosine parts have the same multiple angle).

Example

Find the maximum value of $f(\theta) = \cos 2\theta + \sin 2\theta$ and the smallest possible positive value of θ where this occurs.

First, express this as a single trigonometric function. Any of the four forms can be chosen, but the simplest is generally $k\cos(2\theta-\alpha)$.

Let $f(\theta) = 1\cos 2\theta + 1\sin 2\theta = k\cos(2\theta-\alpha)$
$$= k\cos 2\theta\cos\alpha + k\sin 2\theta\sin\alpha$$

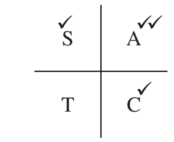
So $k\sin\alpha = 1$
$k\cos\alpha = 1$

Squaring and adding:

$$k^2\sin^2\alpha + k^2\cos^2\alpha = 1^2 + 1^2$$
$$k^2 = 2$$
$$k = \sqrt{2}$$

Dividing:

$$\frac{k\sin\alpha}{k\cos\alpha} = \tan\alpha = \frac{1}{1} = 1$$
$$\tan^{-1}(1) = \frac{\pi}{4}$$



So $\alpha = \dfrac{\pi}{4}$

Hence $\cos 2\theta + \sin 2\theta = \sqrt{2}\cos\left(2\theta - \dfrac{\pi}{4}\right)$

The maximum value of $f(\theta)$ is $\sqrt{2}$. This normally occurs when $\theta = 0$ for $\cos\theta$ and so here

$2\theta - \dfrac{\pi}{4} = 0$
$\Rightarrow 2\theta = \dfrac{\pi}{4}$
$\Rightarrow \theta = \dfrac{\pi}{8}$

There will be an infinite number of maximum points but this is the first one with a positive value of θ.

This method can also be used to solve equations (if no calculator is available).

Example

Solve $\sqrt{3}\cos 2\theta - \sin 2\theta - 1 = 0$ by first expressing $\sqrt{3}\cos 2\theta - \sin 2\theta$ in the form $k\cos(2\theta - \alpha)$.

Let $\sqrt{3}\cos 2\theta - \sin 2\theta = k\cos(2\theta-\alpha)$
$$= k\cos 2\theta\cos\alpha + k\sin 2\theta\sin\alpha$$

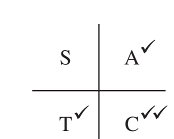
So $k\sin\alpha = -1$
$k\cos\alpha = \sqrt{3}$

Squaring and adding:

$$k^2\sin^2\alpha + k^2\cos^2\alpha = (-1)^2 + \sqrt{3}^2$$
$$k^2 = 4$$
$$k = 2$$

Dividing:

$$\frac{k\sin\alpha}{k\cos\alpha} = \tan\alpha = \frac{-1}{\sqrt{3}}$$
$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$



So $\alpha = 2\pi - \dfrac{\pi}{6}$

$= \dfrac{11\pi}{6}$

Hence $\sqrt{3}\cos 2\theta - \sin 2\theta - 1 = 2\cos\left(2\theta - \dfrac{11\pi}{6}\right) - 1$

The equation becomes

$$2\cos\left(2\theta - \frac{11\pi}{6}\right) - 1 = 0$$

$$\Rightarrow \cos\left(2\theta - \frac{11\pi}{6}\right) = \frac{1}{2}$$

$$\Rightarrow 2\theta - \frac{11\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\Rightarrow 2\theta = \frac{13\pi}{6}, \frac{21\pi}{6}$$

$$\Rightarrow \theta = \frac{13\pi}{12}, \frac{21\pi}{12}$$

There will be two more solutions one period away (π radians).

So $\theta = \dfrac{\pi}{12}, \dfrac{9\pi}{12}, \dfrac{13\pi}{12}, \dfrac{21\pi}{12}$

Exercise 5

1 Express each of these in the form $k\cos(x - \alpha)°$, where $k > 0$, $0° \le \alpha° < 360°$.

a $6\cos x° + 8\sin x°$ b $5\cos x° + 12\sin x°$

c $\cos x° - 3\sin x°$ d $\sin x° - 2\cos x°$

2 Express each of these in the form $k\cos(\theta - \alpha)$, where $k > 0$, $0 \le \alpha < 2\pi$.

a $\sqrt{3}\cos\theta - \sin\theta$ b $\cos\theta - \sin\theta$

c $-\cos\theta - 2\sin\theta$ d $\sqrt{3}\sin\theta - \cos\theta$

3 Express each of these in the form $k\cos(x + \alpha)°$, where $k > 0$, $0° \le \alpha° < 360°$.

a $15\cos x° - 8\sin x°$ b $2.5\cos x° - 3.5\sin x°$

4 Express each of these in the form $k\sin(\theta + \alpha)$, where $k > 0$, $0 \le \alpha < 2\pi$.

a $\sqrt{3}\cos\theta - \sin\theta$ b $\cos\theta - \sin\theta$

5 Express each of these in the form $k\sin(x - \alpha)°$, where $k > 0$, $0° \le \alpha° < 360°$.

a $-\sin x° - 3\cos x°$ b $\sqrt{3}\cos x° - \sin x°$

6 Express each of these as a single trigonometric function.

a $\cos 2x° + \sin 2x°$ b $\cos 3x° - \sqrt{3}\sin 3x°$

c $\sqrt{27}\cos\theta - 3\sin\theta$ d $\sin 30\theta - \cos 30\theta$

7 Without using a calculator, state the maximum and minimum values of each function, and the corresponding values of $x°$, for $0° \le x° < 360°$.

a $f(x) = 5\cos x° - 5\sin x°$

b $f(x) = \sqrt{3}\cos x° - \sin x° + 5$

c $f(x) = 7\sin 2x° - 7\cos 2x° + 1$

8 State the maximum and minimum values of each function, and the corresponding values of θ, for $0 \le \theta < 2\pi$.

a $f(\theta) = \sqrt{8}\sin\theta - 2\cos\theta$

b $f(\theta) = -\sqrt{3}\cos 3\theta - \sin 3\theta$

c $f(\theta) = \sqrt{6}\sin 8\theta - \sqrt{2}\cos 8\theta - 3$

9 By expressing the left-hand side of each equation as a single trigonometric function, solve the equation for $0 \le \theta < 2\pi$.

a $\cos\theta - \sin\theta = -1$ b $\sqrt{3}\cos\theta - \sin\theta = \sqrt{3}$

c $\cos\theta - \sin\theta = 1$ d $\cos 4\theta - \sqrt{3}\sin 4\theta = 1$

10 By expressing the left-hand side of each equation as a single trigonometric function, solve the equation for $0° \le x° < 360°$.

a $3\sin x° - 5\cos x° = 4$ b $8\cos 3x° + 15\sin 3x° = 13$

Review exercise

1 If $\sin x° = \dfrac{1}{2}$, what are possible values for $\cos x°$?

2 For the billiards shot shown in the diagram,

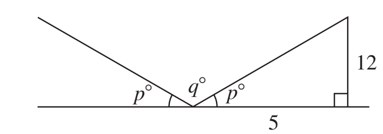
a prove that

i $\sin q° = \sin 2p°$

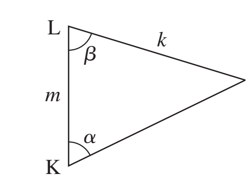
ii $\cos q° = -\cos 2p°$

b Find the value of

i $\sin q°$ ii $\cos q°$



3 Show that, for triangle KLM, $m = \dfrac{k\sin(\alpha + \beta)}{\sin\alpha}$.



4 Prove that $\dfrac{1 + \sin 2\phi}{\cos 2\phi} \equiv \dfrac{\cos\phi + \sin\phi}{\cos\phi - \sin\phi}$.

5 Prove that $2\csc x \equiv \tan\left(\dfrac{x}{2}\right) + \cot\left(\dfrac{x}{2}\right)$.

6 Solve the equation $\cos\theta + \cos 3\theta + \cos 5\theta = 0$ for $0 \le \theta < \pi$.

7 The function f is defined on the domain $[0, \pi]$ by $f(\theta) = 4\cos\theta + 3\sin\theta$.

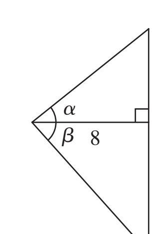
a Express $f(\theta)$ in the form $R\cos(\theta - \alpha)$ where $R > 0$, $0 \le \alpha < 2\pi$.

b Hence, or otherwise, write down the value of θ for which $f(\theta)$ takes its maximum value. [IB May 02 P1 Q12]

8 Find all the values of θ in the interval $[0, \pi]$ that satisfy the equation $\cos\theta = \sin^2\theta$. [IB May 03 P1 Q2]

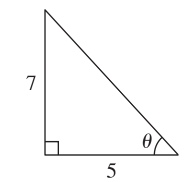
9 Use the fact that $-15° = (45 - 60)°$ to find $\sin(-15)°$.

10 In the following diagram, find $\cos(\alpha + \beta)$.



11 Find csc 22.5° by using a half angle formula.

12 Find $\tan 2\theta$.



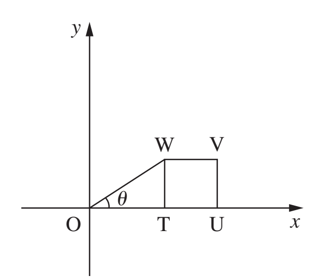
13 Express $\sqrt{48}\cos\theta - 8\sin\theta$ in the form $k\sin(\theta - \alpha)$ where $k > 0$, $0 \leq \alpha < 2\pi$.

14 Express $6\cos 8x° - 6\sin 8x°$ in the form $k\cos(8x - \alpha)°$. Hence solve $6\cos 8x° - 6\sin 8x° + 7 = 1$ for $0° \leq x° < 90°$.

15 Prove that $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 + 3\tan^2\theta}$.

16 K is the point with coordinates $\left(\sin\left(\theta + \frac{\pi}{6}\right), \cos\left(\theta - \frac{\pi}{6}\right)\right)$ and L has coordinates $\left(\sin\left(\theta - \frac{\pi}{6}\right), \cos\left(\theta + \frac{\pi}{6}\right)\right)$. Find, in its simplest form, an expression for the gradient of the line KL.

17 The rotor blade of a helicopter is modelled using the following diagram, where TUVW is a square.



a Show that the area of OTUVW is

$$A = \sin^2\theta + \frac{1}{4}\sin 2\theta$$

b Express the area in the form $k\cos(2\theta - \alpha) + p$.

c Hence find the maximum area of the rotor blade, and the smallest positive value of θ when this maximum occurs.