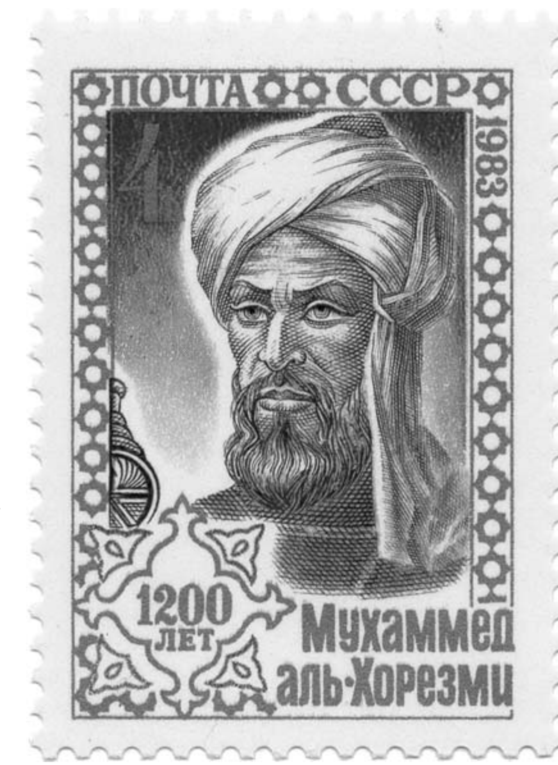
2 Quadratic Equations, Functions and Inequalities

The first reference to quadratic equations appears to be made by the Babylonians in 400 BC, even though they did not actually have the notion of an equation. However, they succeeded in developing an algorithmic approach to solving problems that could be turned into quadratic equations. Most of the problems that the Babylonians worked on involved length, hence they had no concept of a negative answer. The Hindu mathematician, Brahmagupta, undertook more work in the seventh century and he realised that negative quantities were possible and he worked on the idea of letters for unknowns. In the ninth century, in his book *Hisab al-jabr w'al-muqabala*, Al-khwarizmi solved quadratic equations entirely in words. The word "algebra" is derived from the title of his book. It was not until the twelfth century that Abraham bar Hiyya ha-Nasi finally developed a full solution to a quadratic equation.

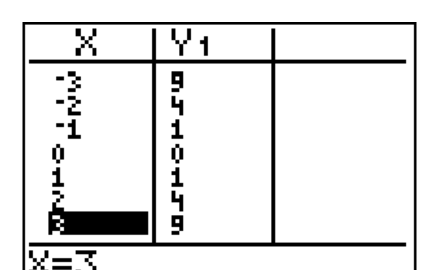


USSR stamp featuring Al-khwarizmi

2.1 Introduction to quadratic functions

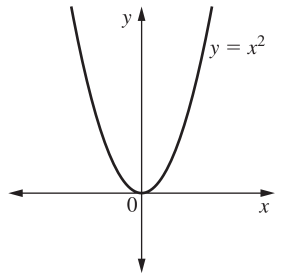
Consider the curve $y = x^2$. What does it look like?

To draw the graph a table of values can be set up on a calculator or drawn on paper.



x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
y	9	4	1	0	1	4	9

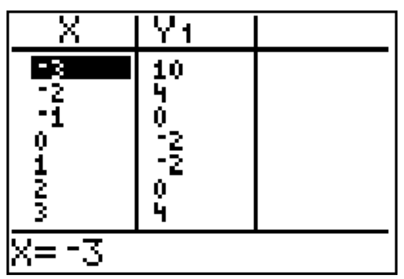
The curve is shown below.



The main features of the curve are:

- It is symmetrical about the *y*-axis.
- The *y*-values are always greater than or equal to zero (there are no negative *y*-values).
- The minimum value is $y = 0$.

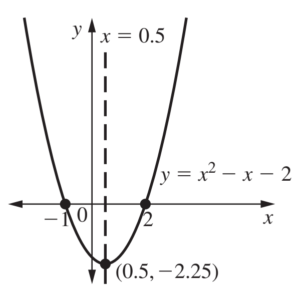
Now consider the curve $y = x^2 - x - 2$. This is the table of values:



X	Y1
-3	10
-2	4
-1	0
0	-2
1	-2
2	0
3	4

X= -3

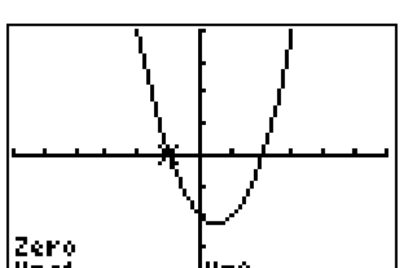
The curve is shown below.

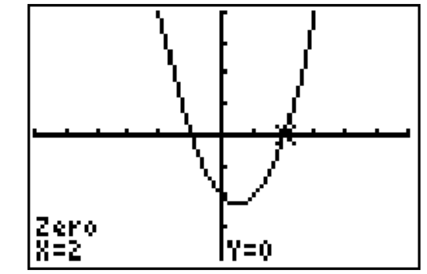


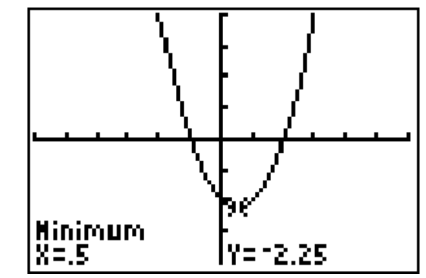
In this case:

- The line of symmetry is $x = 0.5$.
- The curve intersects the *x*-axis at $x = -1$ and $x = 2$.
- The minimum value of the curve is $y = -2.25$.

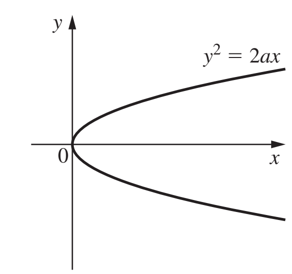
This information can also be found on the calculator, and the displays for this are shown below.







The standard quadratic function is $f(x) = ax^2 + bx + c$ where $a \in \mathbb{R}, a \neq 0$ and $b, c \in \mathbb{R}$. Its graph is known as a **parabola**. However, this is not the only form that produces a parabolic graph. Graphs of the form $y^2 = 2ax$ are also parabolic, and an example is shown below.



Investigation

Sketch the following 12 curves using your calculator:

$y = x^2 - 5x + 6$	$y = x^2 - 4x + 4$	$y = 2x^2 + 5x + 2$
$y = 2x^2 + 5x - 3$	$y = -x^2 - 6x - 8$	$y = -x^2 - 4x - 6$
$y = x^2 + 7x + 9$	$y = -x^2 + 4x - 4$	$y = 4x^2 - 4x + 1$
$y = -x^2 + 3x - 7$	$y = -4x^2 - 12x - 9$	$y = -x^2 + 4x - 3$

Use the graphs to deduce general rules. Think about about the following points:

- When do these curves have maximum turning points? When do they have minimum turning points?
- What is the connection between the *x*-value at the turning point and the line of symmetry?
- What is the connection between the *x*-intercepts of the curve and the line of symmetry?
- Describe the intersection of the curve with the *x*-axis.

The maximum or minimum turning point is the point where the curve turns and has its greatest or least value. This will be looked at in the context of other curves in Chapter 8.

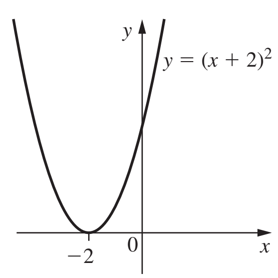
From this investigation, it is possible to deduce that:

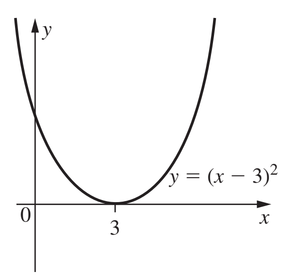
- If the coefficient of x^2 is positive, the curve has a minimum turning point. If it is negative, the curve has a maximum turning point.
- The x-value where the maximum or minimum turning point occurs is also the line of symmetry.
- The line of symmetry is always halfway between the x-intercepts if the curve crosses the x-axis twice.
- There are three possible scenarios for the intersection of the curve with the x-axis:
  - It intersects twice.
  - It touches the x-axis
  - It does not cross or touch the x-axis at all.

In Chapter 1 transformations of curves were introduced. We will now look at three transformations when applied to the quadratic function.

$y = (x + a)^2$

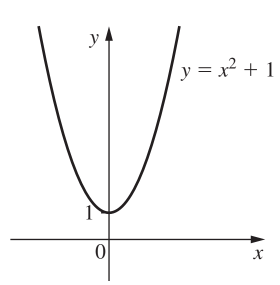
This translates (shifts) the curve $y = x^2$ left by a units if a is positive or right by a units if a is negative. The curve does not change shape; it merely shifts left or right. Examples are shown below.

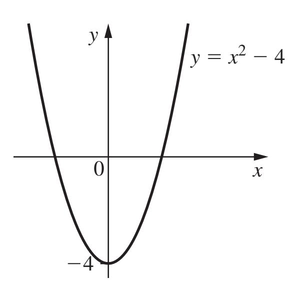




$y = x^2 + a$

This translates (shifts) the curve $y = x^2$ up by a units if a is positive or down by a units if a is negative. Once again, the curve does not change shape; it merely shifts up or down. Examples are shown below.



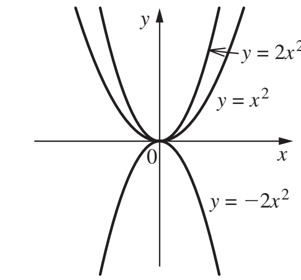


$y = ax^2$

In this case the curve does change shape.

If a is positive this stretches the curve $y = x^2$ parallel to the y-axis by scale factor a. If a is negative the stretch is the same but the curve is also reflected in the x-axis. $y = 2x^2$ and $y = -2x^2$ are shown below.

2 Quadratic Equations, Functions and Inequalities



The parabola shape of a quadratic graph occurs in a number of natural settings. One possible example is the displacement-time graph of a projectile, but it must be remembered that this does not take into account the effect of air resistance. A suspension bridge provides a more realistic example. When a flexible chain is hung loosely between two supports, the shape formed by the chain is a curve called a **catenary**. However, if a load is hung from this chain, as happens in a suspension bridge, then the curve produced is in fact a parabola. Hence the shape of the suspending wires on the Golden Gate Bridge in San Francisco is a parabola.

2.2 Solving quadratic equations

Quadratic equations are equations of the form $ax^2 + bx + c = 0$.

There are effectively two methods of solving a quadratic equation.

Solving a quadratic equation by factorisation

This is where the quadratic equation is rearranged to equal zero, the quadratic expression is factorised into two brackets, and then the values of x for which each bracket equals zero are found. The values of x that make the brackets zero are called the **roots** or **zeros** of the equation and are also the x-intercepts of the curve.

Factorisation was covered in the chapter on presumed knowledge.

Example

Solve the quadratic equation $x^2 - 3x - 10 = 0$.

$$x^2 - 3x - 10 = 0$$
$$\Rightarrow (x - 5)(x + 2) = 0$$
$$\Rightarrow x - 5 = 0 \text{ or } x + 2 = 0$$
$$\Rightarrow x = 5 \text{ or } x = -2$$

Example

Solve the quadratic equation $2x^2 - 5x + 2 = 0$.

$2x^2 - 5x + 2 = 0$

$\Rightarrow (2x - 1)(x - 2) = 0$

$\Rightarrow 2x - 1 = 0$ or $x - 2 = 0$

$\Rightarrow x = \frac{1}{2}$ or $x = 2$

Using the formula to solve a quadratic equation

Some quadratic equations cannot be factorised, but still have solutions. This is when the quadratic formula is used:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a, b and c refer to $ax^2 + bx + c = 0$.

$b^2 - 4ac$ is known as the **discriminant**.

We begin by looking at the previous example, which was solved by factorisation, and show how it can be solved using the quadratic formula.

The formula works for all quadratic equations, irrespective of whether they factorise or not, and will be proved later in the chapter.

Example

Solve the quadratic equation $2x^2 - 5x + 2 = 0$ using the quadratic formula.

In this case, $a = 2$, $b = -5$ and $c = 2$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)}$

$\Rightarrow x = \frac{5 \pm \sqrt{25 - 16}}{4}$

$\Rightarrow x = \frac{5 \pm 3}{4}$

$\Rightarrow x = 2, \frac{1}{2}$

Example

Solve the equation $x^2 + 6x - 10 = 0$ using the quadratic formula.

$x^2 + 6x - 10 = 0$

In this case, $a = 1$, $b = 6$ and $c = -10$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-10)}}{2(1)}$

$\Rightarrow x = \frac{-6 \pm \sqrt{36 + 40}}{2}$

$\Rightarrow x = \frac{-6 \pm \sqrt{76}}{2}$

$\Rightarrow x = -7.36, 1.36$

Exercise 1

1 Find the solutions to the following quadratic equations using factorisation.
  a $x^2 - 5x + 4 = 0$ **b** $x^2 - x - 6 = 0$
  c $2x^2 + 17x + 8 = 0$ **d** $x(x - 1) = x + 3$

2 Find the x-intercepts on the following curves using factorisation.
  a $y = x^2 - 7x + 10$ **b** $y = x^2 - 5x - 24$ **c** $y = 2x^2 + 5x - 12$
  d $y = 6x^2 + 5x - 6$ **e** $y = 3x^2 - 11x + 6$

3 Find the solutions to the following quadratic equations using the quadratic formula.
  a $x^2 - 6x + 6 = 0$ **b** $x^2 - 5x - 5 = 0$ **c** $2x^2 + 7x + 2 = 0$
  d $5x^2 + 9x + 2 = 0$ **e** $3x^2 + 5x - 3 = 0$

4 Find the x-intercepts on the following curves using the quadratic formula.
  a $y = x^2 + 6x + 3$ **b** $y = x^2 - 4x - 9$ **c** $y = 3x^2 + 7x + 3$
  d $y = 2x^2 + 5x - 11$ **e** $y = 1 - 2x^2 - 3x$

2.3 Quadratic functions

We can write quadratic functions in a number of different forms.

Standard form

This is the form $f(x) = ax^2 + bx + c$ where c is the y-intercept because $f(0) = c$. Remember that if a is positive the curve will have a minimum point, and if a is negative the curve will have a maximum turning point. If the curve is given in this form, then to draw it we would usually use a calculator or draw a table of values.

Intercept form

This is the form $f(x) = (ax + b)(cx + d)$ where the quadratic function has been factorised. In this form the x-intercepts are given by $-\frac{b}{a}$ and $-\frac{c}{d}$ since they are found by letting $f(x) = 0$. Knowing the x-intercepts and the y-intercept, there is normally enough information to draw the curve.

Example

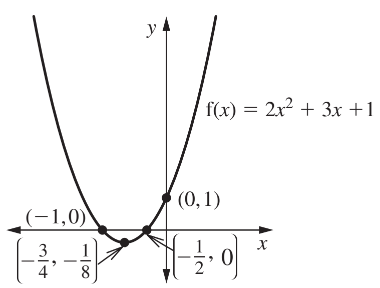
Sketch the curve $f(x) = 2x^2 + 3x + 1$.
The y-intercept is at $f(0) = 1$.
Factorising the function:
$f(x) = (2x + 1)(x + 1)$
So to find the x-intercepts solve $(2x + 1)(x + 1) = 0$.

Hence the x-intercepts are $-\frac{1}{2}$ and -1.

The curve has a positive coefficient of x^2, so it has a minimum turning point.

This point occurs halfway between $-\frac{1}{2}$ and -1, i.e. at $-\frac{3}{4}$.

When $x = -\frac{3}{4}$, $f(x) = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) + 1 = -\frac{1}{8}$. So the minimum point is $\left(-\frac{3}{4}, -\frac{1}{8}\right)$.
A sketch of the graph is shown below.



Turning point form

This is when the function is in the form $f(x) = r(x - p)^2 + q$. The graph is the curve $y = x^2$ which has been translated p units to the right, stretched parallel to the y-axis by scale factor r, then translated q upwards.

In this form the maximum or minimum turning point has coordinates (p,q). This is the reason:

If r is positive, then since $(x - p)^2$ is never negative, the least possible value of $f(x)$ is given when $(x - p)^2 = 0$. Hence $f(x)$ has a minimum value of q, which occurs when $x = p$.

If r is negative, then since $(x - p)^2$ is never negative, the greatest possible value of $f(x)$ is given when $(x - p)^2 = 0$. Hence $f(x)$ has a maximum value of q, which occurs when $x = p$.

Remember that $x = p$ is the line of symmetry of the curve.

Completing the square

Writing a quadratic in the form $r(x \pm p)^2 \pm q$ is known as completing the square.

This is demostrated in the following examples.

This technique will be needed in later chapters.

Example

Write the function $f(x) = x^2 + 6x - 27$ in the form $(x + p)^2 - q$.
We know from the expansion of brackets $(x - a)(x - a)$ that this equals $x^2 - 2ax + a^2$. Hence the coefficient of x is always twice the value of p.
Therefore $f(x) = (x + 3)^2$ plus or minus a number. To find this we subtract 3^2 because it is not required and then a further 27 is subtracted.
Hence $f(x) = (x + 3)^2 - 9 - 27 = (x + 3)^2 - 36$.

Method for completing the square on $ax^2 + bx + c$

1. Take out a, leaving the constant alone: $a\left(x^2 + \frac{b}{a}x\right) + c$

2. Complete the square: $a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c$

3. Multiply out the outer bracket.

4. Tidy up the constants.

Example

Complete the square on the function $f(x) = 2x^2 + 3x + 8$.
In this case the coefficient of x^2 is 2 and we need to make it 1: hence the 2 is factorised out, but the constant is left unchanged.

$f(x) = 2\left(x^2 + \frac{3}{2}x\right) + 8$

Following the same procedure as the example above:

$f(x) = 2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16}\right] + 8$

$f(x) = 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} + 8 = 2\left(x + \frac{3}{4}\right)^2 + \frac{55}{8}$

Example

Complete the square on the function $f(x) = -x^2 + 6x - 11$.
Step 1 $\Rightarrow f(x) = -(x^2 - 6x) - 11$
Step 2 $\Rightarrow f(x) = -[(x - 3)^2 - 9] - 11$
Step 3 $\Rightarrow f(x) = -(x - 3)^2 + 9 - 11$
Step 4 $\Rightarrow f(x) = -(x - 3)^2 - 2$

Example

Complete the square on the function $f(x) = 2x^2 - 6x + 9$ and hence find the maximum or minimum turning point.

$$f(x) = 2x^2 - 6x + 9$$
$$\Rightarrow f(x) = 2(x^2 - 3x) + 9$$
$$\Rightarrow f(x) = 2\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 9$$
$$\Rightarrow f(x) = 2\left(x - \frac{3}{2}\right)^2 - \frac{9}{2} + 9$$
$$\Rightarrow f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{9}{2}$$

The curve will have its least value when $x - \frac{3}{2} = 0$, that is $x = \frac{3}{2}$, and this least value will be $\frac{9}{2}$. Hence the curve has a minimum turning point, which is $\left(\frac{3}{2}, \frac{9}{2}\right)$.

Example

Without using a calculator, sketch the curve $y = 2x^2 + 6x - 8$.
From this form of the curve we note that the y-intercept occurs when $x = 0$ and is therefore $y = -8$.
By turning it into intercept form the x-intercepts can be found.

$$y = 2(x^2 + 3x - 4)$$
$$\Rightarrow y = 2(x + 4)(x - 1)$$

Hence the x-intercepts are when $(x + 4)(x - 1) = 0$
$$\Rightarrow x = -4, x = 1$$

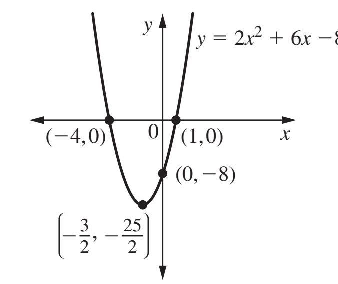
Complete the square to transform it into turning point form:

$$y = 2(x^2 + 3) - 8$$
$$\Rightarrow y = 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4}\right] - 8$$
$$\Rightarrow y = 2\left(x + \frac{3}{2}\right)^2 - \frac{9}{2} - 8$$
$$\Rightarrow y = 2\left(x + \frac{3}{2}\right)^2 - \frac{25}{2}$$

Since the coefficient of x^2 is positive, we know that the curve has a minimum turning point with coordinates $\left(-\frac{3}{2}, -\frac{25}{2}\right)$.

> If the quadratic equation does not factorise, use the formula. If there is no solution the curve is entirely above or below the x-axis.

Hence the curve is:



We could sketch the curve from the intercept form alone, but if the curve does not cut the x-axis the turning point form must be used.

Example

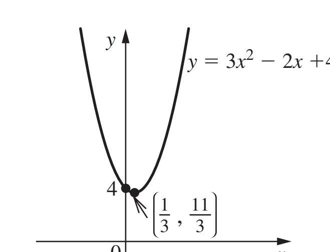
By using a method of completing the square without a calculator, sketch the curve $y = 3x^2 - 2x + 4$.
In this form we can see that the curve cuts the y-axis at (0,4).
To find the x-intercepts it is necessary to solve $3x^2 - 2x + 4 = 0$.

Using the quadratic formula: $x = \dfrac{2 \pm \sqrt{4 - 48}}{6} = \dfrac{2 \pm \sqrt{-44}}{6}$

This gives no real roots and hence the curve does not cut the x-axis.
Hence in this situation the only way to find the turning point is to complete the square.

$$y = 3x^2 - 2x + 4$$
$$\Rightarrow y = 3\left(x^2 - \frac{2}{3}\right) + 4$$
$$\Rightarrow y = 3\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9}\right] + 4$$
$$\Rightarrow y = 3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3} + 4$$
$$\Rightarrow y = 3\left(x - \frac{1}{3}\right)^2 + \frac{11}{3}$$

Since the coefficient of x^2 is positive, we know that the curve has a minimum turning point with coordinates $\left(\frac{1}{3}, \frac{11}{3}\right)$.

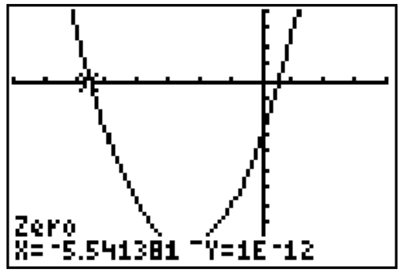


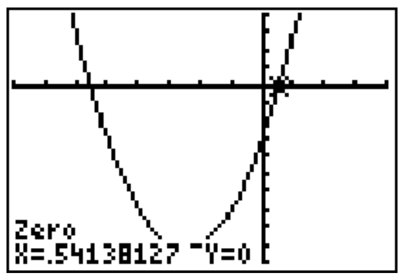
All these curves can be sketched using a calculator and the x-intercepts, y-intercept and the maximum or minimum turning point found.

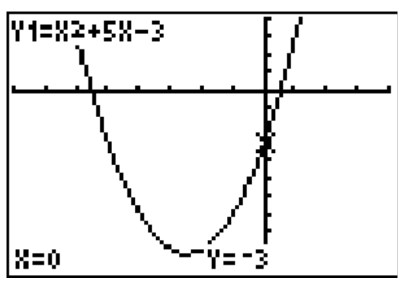
2 Quadratic Equations, Functions and Inequalities

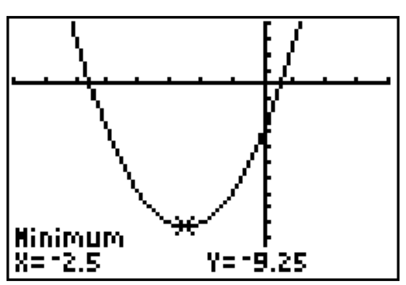
Example

Using a calculator, sketch the curve $y = x^2 + 5x - 3$ showing the coordinates of the minimum turning point and the x- and y-intercepts.









It is also possible to solve quadratic equation using the method of completing the square.

Example

Solve the quadratic equation $x^2 + 6x - 13 = 0$ by completing the square.

$$x^2 + 6x - 13 = 0$$
$$\Rightarrow (x + 3)^2 - 9 - 13 = 0$$
$$\Rightarrow (x + 3)^2 - 24 = 0$$
$$\Rightarrow (x + 3)^2 = 24$$
$$\Rightarrow x + 3 = \pm\sqrt{24}$$
$$\Rightarrow x = -3 \pm 2\sqrt{6}$$

This method is not often used because the answer looks identical to the answer found using the quadratic formula. In fact the formula is just a generalised form of completing the square, and this is how we prove the quadratic formula.

We begin with $ax^2 + bx + c = 0$

$$\Rightarrow a\left(x^2 + \frac{b}{a}\right) + c = 0$$
$$\Rightarrow a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c = 0$$
$$\Rightarrow a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$
$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$
$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} = 0$$
$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$\Rightarrow x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$
$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From here it should be noted that $x = -\frac{b}{2a}$ is the line of symmetry of the curve and the x-value where the maximum or minimum turning point occurs.

Exercise 2

1 Complete the square:
  a $x^2 + 2x + 5$ b $x^2 - 3x + 3$ c $-x^2 + 3x - 5$
  d $3x^2 + 6x - 8$ e $5x^2 + 7x - 3$

2 Complete the square and hence sketch the parabola, showing the coordinates of the maximum or minimum turning point and the x- and y-intercepts.
  a $y = x^2 + 6x + 4$ b $y = x^2 - 4x + 3$ c $y = x^2 + 5x + 2$
  d $y = -x^2 - 4x + 3$ e $y = -x^2 + 8x + 3$ f $y = 2x^2 + 10x - 11$
  g $y = 4x^2 - 3x + 1$ h $y = 3x^2 + 5x + 2$ i $y = -2x^2 + 3x + 4$

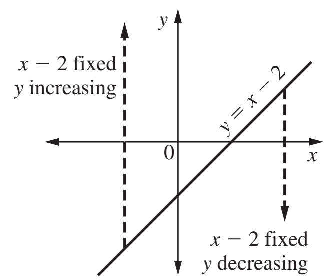
3 Draw each of these on a calculator and identify the maximum or minimum turning point, the x-intercepts and the y-intercept.
  a $y = 2x^2 + 5x - 7$ b $y = -x^2 + 5x - 7$
  c $y = 5x^2 + 6x + 16$ d $y = -3x^2 + 5x + 9$

2.4 Linear and quadratic inequalities

Linear inequalities

The equation $y = mx + c$ represents a straight line. We now need to consider what is meant by the inequalities $y > mx + c$ and $y < mx + c$.

Consider the line $y = x - 2$. At any point on the line the value of y is equal to the value of $x - 2$. What happens when we move away from the line in a direction parallel to the y-axis? Clearly the value of $x - 2$ stays the same, but the value of y will increase if we move in the direction of positive y and decrease if we move in the direction of negative y. Hence for all points above the line $y > x - 2$ and for all points below the line $y < x - 2$. This is shown below.



This argument can now be extended to $y = mx + c$. Hence all points above the line fit the inequality $y > mx + c$ and all points below the line fit the inequality $y < mx + c$.

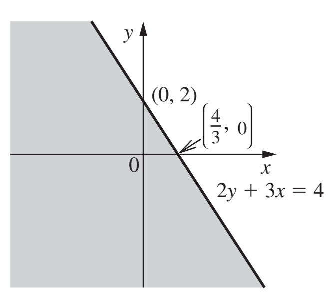
Example

Sketch the region of the x, y plane represented by the inequality $2y + 3x > 4$.

First we consider the line $2y + 3x = 4$.

Rearranging this into the form $y = mx + c$ gives $y = -\frac{3}{2}x + 2$.

This line has gradient $-\frac{3}{2}$ and a y-intercept of 2. Since "greater than" is required, the area above the line is needed. This is shown below.



The shading shows the non-required area. The line is not included.

Now we consider how to solve linear inequalities.

Consider the fact that $10 > 9$. If the same number is added to both sides of the inequality, then the inequality remains true. The same is true if the same number is subtracted from both sides of the inequality.

Now consider multiplication. If both sides are multiplied by 3, this gives $30 > 27$, which is still a true statement. If both sides are multiplied by -3 then $-30 > -27$, which is no longer true. Division by a negative number leads to the same problem.

This demonstrates the general result that whenever an inequality is multiplied or divided by a negative number then the inequality sign must reverse.

In all other ways, linear inequalities are solved in the same way as linear equations. We can avoid this problem by ensuring that the coefficient of x always remains positive.

Example

Find the solution set to $2(x - 1) > 3 - 4(x + 1)$.

$$2(x - 1) > 3 - 4(1 - x)$$
$$\Rightarrow 2x - 2 > 3 - 4 + 4x$$
$$\Rightarrow -2x > 1$$
$$\Rightarrow x < -\frac{1}{2}$$

Alternatively:

$$2(x - 1) > 3 - 4(1 - x)$$
$$\Rightarrow 2x - 2 > 3 - 4 + 4x$$
$$\Rightarrow -1 > 2x$$
$$\Rightarrow x < -\frac{1}{2}$$

Quadratic inequalities

Any inequality that involves a quadratic function is called a **quadratic inequality**. Quadratic inequalities are normally solved by referring to the graph. This is best demonstrated by example.

Example

Find the solution set that satisfies $x^2 + 2x - 15 > 0$.

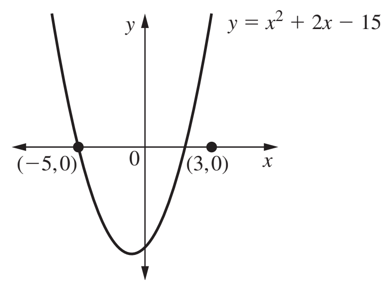
Begin by sketching the curve $y = x^2 + 2x - 15$. This is essential in these questions. However, we are interested only in whether the curve has a maximum or minimum point and where the x-intercepts are.

Since the coefficient of x^2 is positive, the curve has a minimum point and the x-intercepts can be found by factorisation.

$$x^2 + 2x - 15 = 0$$
$$\Rightarrow (x + 5)(x - 3) = 0$$
$$\Rightarrow x = -5, x = 3$$

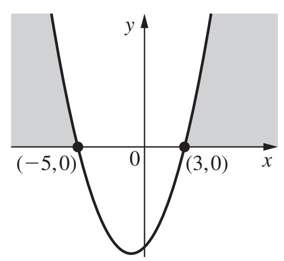
If the quadratic formula does not factorise, use the formula or a graphing calculator.

This curve is shown below.



The question now is "When is this curve greater than zero?" The answer to this is when the curve is above the x-axis.

This is shown below.



The shading shows the non-required area. The line is not included.

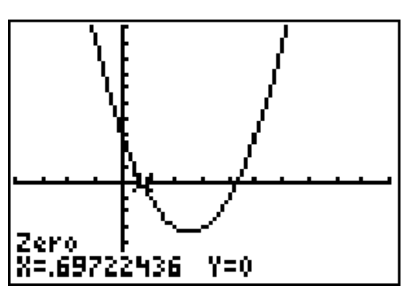
The answer needs to be represented as two inequalities.

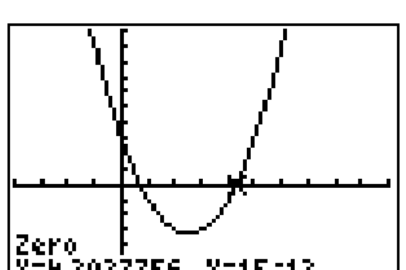
Hence the solution set is $x < -5$ and $x > 3$

This can also be done using a calculator to draw the graph, finding the x-intercepts and then deducing the inequalities.

Example

Using a calculator, find the solution set to $x^2 - 5x + 3 \geq 0$.
The calculator screen dump is shown below.





From this we can deduce that the solution set is $x \leq 0.697, x \geq 4.30$.

Example

Find the solution set that satisfies $-x^2 + x + 6 > 0$.

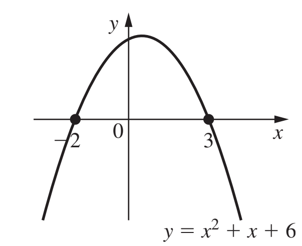
Consider the curve $y = -x^2 + x + 6$. Since the coefficient of x^2 is negative, the curve has a maximum turning point.

Solving $-x^2 + x + 6 = 0$

$\Rightarrow (-x + 3)(x + 2) = 0$

$\Rightarrow x = 3, x = -2$

The curve is shown below.



Since the question asks when the curve is greater than zero, we need to know when it is above the x-axis. Hence the solution set is $-2 < x < 3$.

In this case the solution can be represented as a single inequality.

Exercise 3

1 Show these inequalities on diagrams.

a $y > 2x - 5$ **b** $y < -3x + 8$

c $x + 2y \geq 5$ **d** $3x - 4y + 5 \leq 0$

2 Without using a calculator, find the range(s) of values of x that satisfy these inequalities.

a $x + 5 > 4 - 3x$ **b** $7x - 5 < 2x + 5$

c $-2(3x - 1) - 4(x - 2) \leq 12$ **d** $(x - 3)(x + 5) > 0$

e $(x - 6)(5 - x) < 0$ **f** $(2x + 1)(3 - 4x) \leq 0$

g $2x^2 - 13x + 21 \leq 0$ **h** $x^2 + 7x + 12 > 0$

i $x^2 + 3x + 2 < 0$ **j** $-x^2 - x + 6 \geq 0$

k $x^2 + 12x + 4 \leq -x^2 - 5x - 4$ **l** $4 + 11x + 6x^2 < 0$

3 Using a calculator, find the range(s) of values of x that satisfy these inequalities.

a $x^2 > 6x - 4$ **b** $x^2 - 5x - 5 > 0$

c $-4x^2 + 7x - 1 < 0$ **d** $2x^2 + 4x \leq 3 - x^2 - x$

e $4x^2 \leq 4x + 1$

2.5 Nature of roots of quadratic equations

We now need to take a more in-depth look at quadratic equations.

Consider the following equations and what happens when they are solved using the formula.

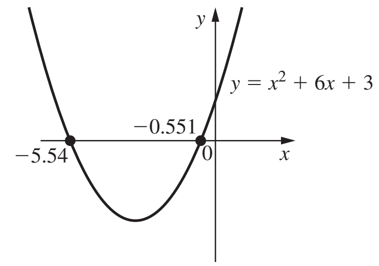
a $x^2 + 6x + 3 = 0$

$$\Rightarrow x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(3)}}{2(1)}$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36 - 12}}{2}$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{24}}{2}$$

$$\Rightarrow x = -0.551, -5.54$$

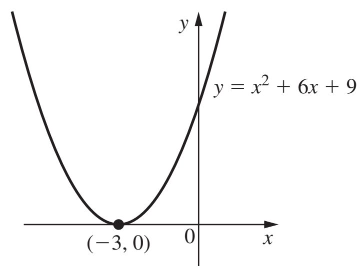


b $x^2 + 6x + 9 = 0$

$$\Rightarrow x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(9)}}{2(1)}$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36 - 36}}{2}$$

$$\Rightarrow x = \frac{-6}{2} = -3$$

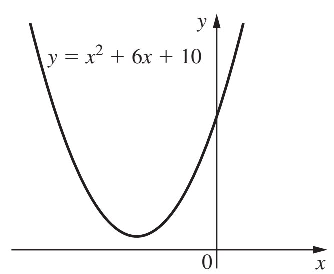


c $x^2 + 6x + 10 = 0$

$$\Rightarrow x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(10)}}{2(1)}$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36 - 40}}{2}$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{-4}}{2}$$



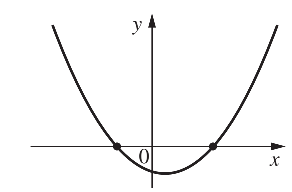
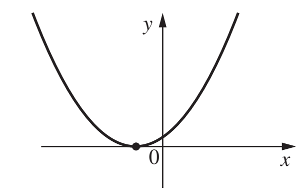
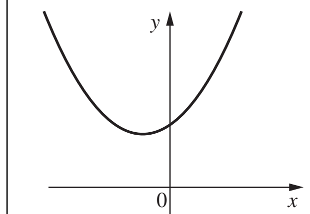
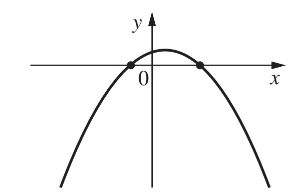
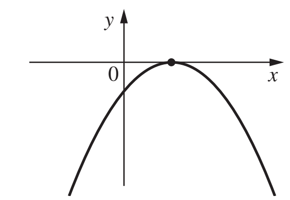
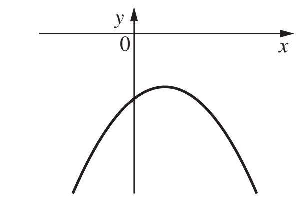
This suggests that the roots of quadratic equations can be classified into three categories and that there are conditions for each category.

In case a) we see that the discriminant is greater than zero, and since the square root exists the $\pm$ produces two different roots. Hence case a) is two real distinct roots, and the condition for it to happen is $b^2 - 4ac > 0$.

In case b) we see that the discriminant is equal to zero so there is only one repeated root, which is $x = -\frac{b}{2a}$. Hence case b) is two real equal roots, and the condition for it to happen is $b^2 - 4ac = 0$. In this case the maximum or minimum turning point is on the x-axis.

In case c) we see that the discriminant is less than zero and so there are no real answers. There are roots to this equation, which are called **complex roots**, but these will be met formally in Chapter 17. Hence case c) is no real roots and the condition for it to happen is $b^2 - 4ac < 0$.

A summary of this is shown in the table below.

$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
		
		

If a question talks about the condition for real roots then $b^2 - 4ac \geq 0$ is used.

Example

Determine the nature of the roots of $x^2 - 3x + 4 = 0$.

In this case $b^2 - 4ac = 9 - 4 = 5$.

Hence $b^2 - 4ac > 0$ and the roots of the equation are real and distinct.

Example

If $ax^2 - 8x + 2 = 0$ has a repeated root, find the value of a.

This is an alternative way of asking about the conditions for real equal roots.

For this equation to have a repeated root, $b^2 - 4ac = 0$.

$$\Rightarrow 64 - 8a = 0$$

$$\Rightarrow a = 8$$

Example

Show that the roots of $2ax^2 + (a + b)x + 3b = 0$ are real for all values of a and b $(a, b \in \mathbb{R})$.

The condition for real roots is $b^2 - 4ac \geq 0$.

$$\Rightarrow (a + b)^2 - 6ab \geq 0$$

$$\Rightarrow a^2 + 2ab + b^2 - 6ab \geq 0$$

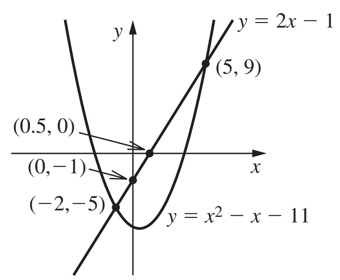
$$\Rightarrow a^2 - 4ab + 4b^2 \geq 0$$

$$\Rightarrow (a - 2b)^2 \geq 0$$

Now the square of any number is always either positive or zero and hence the roots are real irrespective of the values of a and b.

An application of the nature of roots of quadratic equations

Consider the straight line $y = 2x - 1$ and the parabola $y = x^2 - x - 11$. In the figure below we see that the line intersects the curve in two places.



To find the x-coordinates of this point of intersection, we solve the equations simultaneously.

$\Rightarrow 2x - 1 = x^2 - x - 11$

$\Rightarrow x^2 - 3x - 10 = 0$

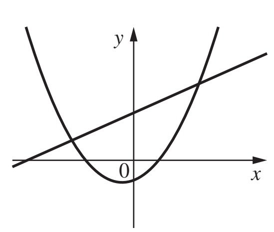
This is the resulting quadratic equation.

$\Rightarrow (x - 5)(x + 2) = 0$

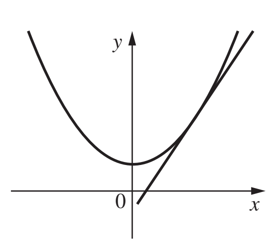
$\Rightarrow x = 5, x = -2$

Hence what can be called the resulting quadratic equation has two real distinct roots. This gives a method of finding the different conditions for which a line and a parabola may or may not intersect. There are three possible cases.

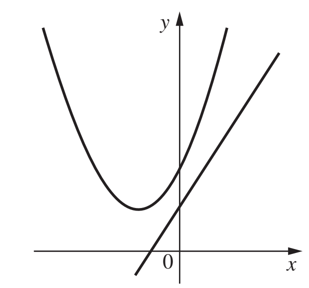
If the parabola and the straight line intersect then there are two roots, and hence this is the case of two real different roots i.e. $b^2 - 4ac > 0$ for the resulting quadratic equation. This is shown below.



If the straight line is a tangent to the parabola (it touches the curve at only one point), then there is one root which is the point of contact and hence this is the case of real, equal roots i.e. $b^2 - 4ac = 0$ for the resulting equation. This is shown below.



If the parabola and the straight line do not intersect then there are no real roots, and hence this is the case of no real roots i.e. $b^2 - 4ac < 0$ for the resulting quadratic equation. This is shown below.



Example

Prove that $y = 4x - 9$ is a tangent to $y = 4x(x - 2)$.

If this is true, then the equation $4x - 9 = 4x(x - 2)$ should have real equal roots, as there is only one point of contact.

$\Rightarrow 4x^2 - 12x + 9 = 0$

In this case $b^2 - 4ac = 144 - 144 = 0$.

Hence it does have real equal roots and $y = 4x - 9$ is a tangent to $y = 4x(x - 2)$.

Exercise 4

1 Determine the nature of the roots of the following equations, but do not solve the equations.

a $x^2 - 3x + 4 = 0$ **b** $2x^2 - 4x - 7 = 0$ **c** $3x^2 - 6x + 4 = 0$

d $-2x^2 + 7x + 2 = 0$ **e** $4x^2 - 4x + 1 = 0$ **f** $x^2 + 1 = 3x - 4$

2 For what values of p is $4x^2 + px + 49 = 0$ a perfect square?

3 Find the value of q if $2x^2 - 3x + q = 0$ has real equal roots.

4 Prove that $qx^2 + 3x - 6 - 4q = 0$ will always have real roots independent of the value of q.

5 Find a relationship between a and b if the roots of $2abx^2 + x\sqrt{a - b} + b^2 - 2a = 0$ are equal.

6 If x is real and $s = \dfrac{4x^2 + 3}{2x - 1}$, prove that $s^2 - 4s - 12 \geq 0$.

7 Find the values of p for which the expression $2p - 3 + 4px - px^2$ is a perfect square.

8 If $x^2 + (3 - 4r)x + 6r^2 - 2 = 0$, show that there is no real value of r.

9 Find the value of m for which the curve $y = 8mx^2 + 3mx + 1$ touches the x-axis.

10 Prove that $y = x - 3$ is a tangent to the curve $y = x^2 - 5x + 6$.

11 For each part of this question, which of the following statements apply?

i The straight line is a tangent to the curve.
ii The straight line cuts the curve in two distinct points.
iii The straight line neither cuts nor touches the curve.

a Curve: $y = 3x^2 - 4x - 2$
Line: $y = x - 3$

b Curve: $y = 7x(x - 1)$
Line: $y + 2x + 1 = 0$

c Curve: $y = 9x^2 - 3x + 10$
Line: $y = 3(x + 3)$

d Curve: $y = (3x - 4)(x + 1)$
Line: $y + 10x + 11 = 0$

Review exercise

1 Express $x(4 - x)$ as the difference of two squares.

2 Given that $y = x^2 - 2x - 3$ $(x \in \mathbb{R})$, find the set of values of x for which $y < 0$. [IB May 87 P1 Q9]

3 Consider the equation $(1 + 2k)x^2 - 10x + k - 2 = 0$, $k \in \mathbb{R}$. Find the set of values of k for which the equation has real roots. [IB Nov 03 P1 Q13]

4 Solve the following simultaneous equations.

i $y - x = 2$
$2x^2 + 3xy + y^2 = 8$

ii $x + y = 9$
$x^2 - 3xy + 2y^2 = 0$

5 The equation $kx^2 - 3x + (k + 2) = 0$ has two distinct real roots. Find the set of possible values of k. [IB May 01 P1 Q18]

6 For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$? [IB Nov 00 P1 Q13]

7 Prove that if $x^2 > k(x - 2)$ for all real x, then $0 < k < 8$.

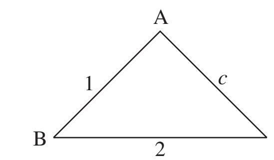
8 By letting $y = x^{\frac{1}{4}}$ find the values of x for which $x^{\frac{1}{4}} - 2x^{-\frac{1}{4}} = 1$.

9 Knowing that the values of x satisfying the equation $2x^2 + kx + k = 0$ are real numbers, determine the range of possible values of $k \in \mathbb{R}$. [IB Nov 91 P1 Q13]

10 Express $(2 - x)(x - 5)$ in the form $a - (x - b)^2$, where a and b are constants. State the coordinates of the maximum point on the graph of $y = (2 - x)(x - 5)$ and also state what symmetry the graph has.

11 William's father is two years older than his mother and his mother's age is the square of his own. The sum of all three ages is 80 years. How old is William?

12



The diagram above shows a triangle ABC, in which $AB = 1$ unit, $BC = 2$ units and $AC = c$ units. Find an expression for cos C in terms of c.

Given that $\cos C > \frac{4}{5}$, show that $5c^2 + 4c + 3 < 0$. Find the set of values of c which satisfy this inequality.