18 Mathematical Induction

Abu Bekr ibn Muhammad ibn al-Husayn Al-Karaji was born on 13 April 953 in Baghdad, Iraq and died in about 1029. His importance the field of mathematics is debated by historians and mathematicians. Some consider that he only reworked previous ideas, while others see him as the first person to use arithmetic style operations with algebra as opposed to geometrical operations.

In his work, *Al-Fakhri*, Al-Karaji succeeded in defining $x,\ x^2, x^3, \ldots$ and $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$ and gave rules for finding the products of any pair without reference to geometry. He was close to giving the rule $x^n x^m = x^{m+n}$ for all integers n and m

but just failed because he did not define $x^0 = 1$.

In his discussion and demonstration of this work Al-Karaji used a form of mathematical induction where he proved a result using the previous result and noted that this process could continue indefinitely. As we will see in this chapter, this is not a full proof by induction, but it does highlight one of the major principles.

Al-Karaji used this form of induction in his work on the binomial theorem, binomial coefficients and Pascal's triangle. The table shown is one that Al-Karaji used, and is actually Pascal's triangle in its side.

He also worked on the sums of the first n natural numbers, the squares of the first n natural numbers and the cubes of these numbers, which we introduced in Chapter 6.

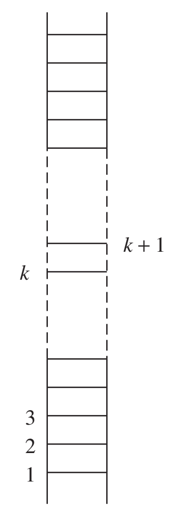
col 1	col 2	col 3	col 4	col 5	· · ·
1	1	1	1	1	· · ·
1	2	3	4	5	· · ·
	1	3	6	10	· · ·
		1	4	10	· · ·
			1	5	· · ·
				1	· · ·

18.1 Introduction to mathematical induction

Mathematical induction is a method of mathematical proof. Most proofs presented in this book are direct proofs – that is proofs where one step leads directly from another to the required result. However, there are a number of methods of indirect proof including proof by contradiction, proof by contrapositive and proof by mathematical induction. In this curriculum, we only consider proof by induction for positive integers.

Mathematical induction is based on the idea of proving the next step to be true if the previous one is true. If the result is true for an initial value, then it is true for all values. This is demonstrated by the following metaphor.

Consider a ladder that is infinite in one direction. We want to prove that the ladder is completely safe, that is each rung on the ladder is sound.



First, test the bottom rung on the ladder and check that it is sound. Then assume that a rung on the ladder, somewhere further up, is also sound. Call this the kth rung. Using this assumption, show that the next rung up, the $(k + 1)$th rung, is also sound **if** the assumption is true. Since we know the first rung is sound, we can now say the second one is sound. As the second one is sound, the third one is sound and so on. So the whole ladder is safe.

Method for mathematical induction

1. Prove the result is true for an initial value (normally $n = 1$).
2. Assume the result to be true for another value, $n = k, k > 1$, stating this result.
3. Consider the case for $n = k + 1$, writing down the goal – the required form.
4. Using the assumption, show that the result is then true for $n = k + 1$.
5. Communicate why this proves the result using mathematical induction.

Example

Prove $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}\ \forall n \in \mathbb{Z}^+$ by mathematical induction.

Remembering the meaning of this notation, we know that $\forall n \in \mathbb{Z}^+$ means that we need to prove it is true for all positive integers, i.e. $n \geq 1, n \in \mathbb{Z}$.

18 Mathematical Induction

1 Prove the result is true for $n = 1$.
It is important to show this very clearly (even though it is often obvious).

$$\text{LHS} = \sum_{r=1}^{1} r = 1 \qquad \text{RHS} = \frac{1(2)}{2} = 1$$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result is true for $n = k, k > 1, k \in \mathbb{Z}$,

i.e. $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$

3 Now consider the result for $n = k + 1$. We want to show that

$$\sum_{r=1}^{k+1} r = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

4 For $n = k + 1$,

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1)$$

Adding on the $(k + 1)$th term.

$$= \frac{k(k+1)}{2} + (k+1)$$

We are using the assumption here.

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

This communication is identical for virtually all induction proofs. It is worth learning its form.

Example

Prove that $\sum_{r=1}^{n} 3r^2 - 5r = n(n+1)(n-2)\ \forall n \in \mathbb{Z}^+$ by mathematical induction.

We cannot use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ here as we are being asked to prove it by mathematical induction.

1 For $n = 1$,

$$\text{LHS} = \sum_{r=1}^{1} 3r^2 - 5r = 3(1)^2 - 5(1) = 3 - 5 = -2 \qquad \text{RHS} = 1(1+1)(1-2) = 1(2)(-1) = -2$$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$,

i.e. $\sum_{r=1}^{k} 3r^2 - 5r = k(k+1)(k-2)$

3 Consider $n = k + 1$. We want to show that

$\sum_{r=1}^{k+1} 3r^2 - 5r = (k+1)(k+1+1)(k+1-2) = (k+1)(k+2)(k-1)$

4 For $n = k + 1$,

$$\sum_{r=1}^{k+1} 3r^2 - 5r$$
$$= \sum_{r=1}^{k}(3r^2 - 5r) + 3(k+1)^2 - 5(k+1) \quad \text{Adding on the } (k+1)\text{th term.}$$
$$= k(k+1)(k-2) + 3(k+1)^2 - 5(k+1) \quad \text{We are using the assumption here.}$$
$$= (k+1)[k(k-2) + 3(k+1) - 5]$$
$$= (k+1)[k^2 + k - 2]$$
$$= (k+1)(k+2)(k-1)$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Example

Prove that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1} \forall n \in \mathbb{Z}^+$ by mathematical induction.

1 For $n = 1$,

LHS $= \sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1(2)} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$,

i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

3 Consider $n = k + 1$. We want to show that

$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+1+1} = \frac{k+1}{k+2}$

4 For $n = k + 1$,

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)}$$
$$= \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)} \quad \text{Adding on the } (k+1)\text{th term.}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{We are using the assumption here.}$$
$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Example

Prove that $\sum_{r=1}^{n} 3^r = \frac{3}{2}(3^n - 1) \forall n \in \mathbb{Z}^+$ by mathematical induction.

1 For $n = 1$,

LHS $= \sum_{r=1}^{1} 3^r = 3^1 = 3$

RHS $= \frac{3}{2}(3^1 - 1) = \frac{3}{2}(2) = 3$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$,

i.e. $\sum_{r=1}^{k} 3^r = \frac{3}{2}(3^k - 1)$

3 Consider $n = k + 1$. We want to show that $\sum_{r=1}^{k+1} 3^r = \frac{3}{2}(3^{k+1} - 1)$

4 For $n = k + 1$,

$$\sum_{r=1}^{k+1} 3^r$$
$$= \sum_{r=1}^{k} 3^r + 3^{k+1} \quad \text{Adding on the } (k+1)\text{th term.}$$
$$= \frac{3}{2}(3^k - 1) + 3^{k+1} \quad \text{Using the assumption.}$$

$$= \frac{3}{2} \cdot 3^k - \frac{3}{2} + 3.3^k$$

$$= 3^k\left(\frac{3}{2} + 3\right) - \frac{3}{2}$$

$$= 3^k\left(\frac{9}{2}\right) - \frac{3}{2}$$

$$= \frac{1}{2}(9.3^k - 3)$$

$$= \frac{3}{2}(3.3^k - 1)$$

$$= \frac{3}{2}(3^{k+1} - 1)$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

It can be seen from these examples that sigma notation is very useful when proving a result by induction.

Example

Prove that $1.4 + 2.5 + 3.6 + \ldots + n(n + 3) = \frac{1}{3}n(n + 1)(n + 5)\ \forall n \in \mathbb{Z}^+$.

It is simpler to express the LHS using sigma notation. Hence the result becomes $\sum_{r=1}^{n} r(r + 3) = \frac{1}{3}n(n + 1)(n + 5)$.

1 For $n = 1$,

$$\text{LHS} = \sum_{r=1}^{1}(1)(1 + 3) = 4 \qquad \text{RHS} = \frac{1}{3}(1)(1 + 1)(1 + 5) = \frac{1}{3}(2)(6) = 4$$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$,

i.e. $\sum_{r=1}^{k} r(r + 3) = \frac{1}{3}k(k + 1)(k + 5)$

3 Consider $n = k + 1$. We want to show that

$$\sum_{r=1}^{k+1} r(r + 3) = \frac{1}{3}(k + 1)(k + 1 + 1)(k + 1 + 5)$$

$$= \frac{1}{3}(k + 1)(k + 2)(k + 6)$$

4 For $n = k + 1$,

$$\sum_{r=1}^{k+1} r(r + 3)$$

$$= \sum_{r=1}^{k} r(r + 3) + (k + 1)(k + 4)$$ Adding on the $(k + 1)$th term.

$$= \frac{1}{3}k(k + 1)(k + 5) + (k + 1)(k + 4)$$ Using the assumption.

$$= \frac{1}{3}(k + 1)[k(k + 5) + 3(k + 4)]$$

$$= \frac{1}{3}(k + 1)[k^2 + 8k + 12]$$

$$= \frac{1}{3}(k + 1)(k + 2)(k + 6)$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Exercise 1

Prove these results $\forall n \in \mathbb{Z}^+$ by mathematical induction.

1 $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$

2 $\sum_{r=1}^{n} 2r - 1 = n^2$

3 $\sum_{r=1}^{n} 3r + 4 = \frac{11}{2}n(3n + 1)$

4 $\sum_{r=1}^{n} 5r - 2 = \frac{1}{2}n(5n + 1)$

5 $\sum_{r=1}^{n} 8 - 3r = \frac{1}{2}n(1 - 3n)$

6 $\sum_{r=1}^{n} 4r^2 - 3 = \frac{4}{3}n(4n^2 + 6n - 9)$

7 $\sum_{r=1}^{n} 6 + 2r - r^2 = 6n(41 + 3n - n^2)$

8 $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n + 1)^2$

9 $\sum_{r=1}^{n} (2r - 1)^3 = n^2(2n^2 - 1)$

10 $\sum_{r=1}^{n} r(r + 1) = \frac{1}{3}(n + 1)(n + 2)$

11 $\sum_{r=1}^{n} r(r + 1)(r + 2) = \frac{1}{4}n(n + 1)(n + 2)(n + 3)$

12 $\sum_{r=1}^{n} (2r)^2 = \frac{2}{3}n(n + 1)(2n + 1)$

13 $\sum_{r=1}^{n} 4^r = \frac{4}{3}(4^n - 1)$

14 $\sum_{r=1}^{n} \frac{1}{r(r + 1)(r + 2)} = \frac{n(n + 3)}{4(n + 1)(n + 2)}$

15 $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \left(\frac{1}{2}\right)^n(n + 2)$

16 $4 + 5 + 6 + \ldots + (n + 3) = \frac{1}{2}n(n + 7)$

17 $5 + 3 + 1 + \ldots + (7 - 2n) = n(6 - n)$

18 $3 + 6 + 11 + \ldots + (n^2 + 2) = \frac{1}{6}n(2n^2 + 3n + 13)$

19 $1.2 + 2.3 + 3.4 + \ldots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

20 $-4 + 0 + 6 + \ldots + (n - 2)(n + 3) = \frac{1}{3}n(n^2 + 3n - 16)$

18.2 Proving some well-known results

So far we have concentrated on proving results that involve sigma notation. However, mathematical induction can be used to prove results from a variety of mathematical spheres. These include results from calculus, complex numbers and matrices as well as algebra.

In earlier chapters, it was stated that proofs would be provided using mathematical induction for the binomial theorem and de Moivre's theorem. In this syllabus, knowledge of the proof of de Moivre's theorem is expected but not for the binomial theorem.

Proof of de Moivre's theorem using mathematical induction

This was proved in Chapter 17 using calculus, but this method must also be known.

Prove de Moivre's theorem for all positive integers, i.e.
$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$

1 For $n = 1$,

$\text{LHS} = (\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta$

$\text{RHS} = \cos(1\theta) + \mathrm{i}\sin(1\theta) = \cos\theta + \mathrm{i}\sin\theta$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$, i.e. $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$

3 Consider $n = k + 1$. We want to show that
$(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = \cos(k + 1)\theta + \mathrm{i}\sin(k + 1)\theta$

Substituting $(k + 1)$ for k in the result.

4 For $n = k + 1$,

$(\cos\theta + \mathrm{i}\sin\theta)^{k+1}$
$= (\cos\theta + \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta)^k$
$= (\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta)$
$= \cos\theta\cos k\theta + \mathrm{i}\sin k\theta\cos\theta + \mathrm{i}\sin\theta\cos k\theta + \mathrm{i}^2\sin\theta\sin k\theta$
$= \cos\theta\cos k\theta - \sin\theta\sin k\theta + \mathrm{i}(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$
$= \cos(\theta + k\theta) + \mathrm{i}(\sin(k\theta + \theta))$
$= \cos(k + 1)\theta + \mathrm{i}\sin(k + 1)\theta$

which is the required form.

We are multiplying the result for $n = k$ by $(\cos\theta + \mathrm{i}\sin\theta)$ to get the result for $n = k + 1$.

Using the assumption.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

This can be extended to negative integers by considering $n = -m$ where m is a positive integer.

Proof of the binomial theorem for positive integer powers

Prove the binomial theorem, i.e. $(x + y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$.

1 For $n = 1$,

$\text{LHS} = (x + y)^1 = x + y$

$$\text{RHS} = \sum_{r=0}^{1}\binom{1}{r}x^{1-r}y^r = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x + y$$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$, i.e.

$$(x + y)^k = \sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^r = \binom{k}{0}x^ky^0 + \binom{k}{1}x^{k-1}y^1 + \binom{k}{2}x^{k-2}y^2 + \ldots + \binom{k}{r}x^{k-r}y^r + \ldots + \binom{k}{k}x^0y^k$$

3 Consider $n = k + 1$. We want to show that $(x + y)^{k+1} = \sum_{r=0}^{k+1}\binom{k+1}{r}x^{k+1-r}y^r$

4 For $n = k + 1$,

$$(x + y)^{k+1}$$
$$= (x + y)(x + y)^k$$
$$= (x + y)\sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^r$$
$$= (x + y)\left[\binom{k}{0}x^ky^0 + \binom{k}{1}x^{k-1}y^1 + \binom{k}{2}x^{k-2}y^2 + \ldots + \binom{k}{r-1}x^{k-r+1}y^{r-1} + \binom{k}{r}x^{k-r}y^r + \ldots + \binom{k}{k}x^0y^k\right]$$
$$= \binom{k}{0}x^{k+1}y^0 + \binom{k}{1}x^ky^1 + \binom{k}{2}x^{k-1}y^2 + \ldots + \binom{k}{r-1}x^{k-r}y^{r-1} + \binom{k}{r}x^{k+1-r}y^r + \ldots + \binom{k}{k}x^1y$$
$$+ \binom{k}{0}x^ky^1 + \binom{k}{1}x^{k-1}y^2 + \binom{k}{2}x^{k-2}y^3 + \ldots + \binom{k}{r-1}x^{k-r+1}y^r + \binom{k}{r}x^{k-r}y^{r+1} + \ldots + \binom{k}{k}x^0y^{k+1}$$
$$= \binom{k}{0}x^{k+1} + \left[\binom{k}{1} + \binom{k}{0}\right]x^ky^1 + \left[\binom{k}{2} + \binom{k}{1}\right]x^{k-1}y^2 + \ldots$$
$$+ \left[\binom{k}{r} + \binom{k}{r-1}\right](x^{k+1-r}y^r) + \ldots + \left[\binom{k}{k} + \binom{k}{k-1}\right]xy^k + \binom{k}{k}y^{k+1}$$

We can now use the result that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ (which was proved in Chapter 6).

So the general term becomes $\left[\binom{k}{r} + \binom{k}{r-1}\right](x^{k+1-r}y^r)$

$$= \binom{k+1}{r}x^{k+1-r}y^r$$

The expansion is therefore

$$\binom{k+1}{0}x^{k+1} + \binom{k+1}{1}x^ky^1 + \binom{k+1}{2}x^{k-1}y^2 + \ldots + \binom{k+1}{r}(x^{k+1-r}y^r) + \ldots$$
$$+ \binom{k+1}{k}xy^k + \binom{k+1}{k+1}y^{k+1}$$
$$= \sum_{r=0}^{k+1}\binom{k+1}{r}x^{k+1-r}y^r$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

We also use mathematical induction to prove divisibility. This is demonstrated in the example below.

Example

Prove that $3^{2n} + 7$ is divisible by 8 for $n \in \mathbb{Z}^+$.

This can be restated as $3^{2n} + 7 = 8p, p \in \mathbb{N}$.

1 For $n = 1$,

$$3^{2n} + 7 = 3^2 + 7$$
$$= 16$$

As 8 is a factor of 16, or $16 = 8 \times 2$, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$, i.e. $3^{2k} + 7 = 8p, p \in \mathbb{N}$.

3 Consider $n = k + 1$. We want to show that $3^{2(k+1)} + 7 = 8t, t \in \mathbb{N}$.

4 For $n = k + 1$,

$$3^{2(k+1)} + 7 = 3^{2k+2} + 7$$
$$= 3^2 3^{2k} + 7$$
$$= 9.3^{2k} + 7$$
$$= 9(3^{2k} + 7 - 7) + 7$$

This allows us to use the assumption.

$$= 9(8p - 7) + 7$$
$$= 9.8p - 63 + 7$$
$$= 9.8p - 56$$
$$= 8(9p - 7)$$

Since $(9p - 7) \in \mathbb{N}$, we can say that $t = 9p - 7$ which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

There are other algebraic results that we can prove using mathematical induction, as exemplified below.

Example

Prove that $2^n > 2n + 1$ for all $n \geq 3, n \in \mathbb{N}$ using mathematical induction.

1 Notice here that the initial value is not $n = 1$.

For $n = 3$,

LHS $= 2^3 = 8$ RHS $= 2(3) + 1 = 7$

Since LHS > RHS, the result is true for $n = 3$.

2 Assume the result to be true for $n = k, k > 3$, i.e. $2^k > 2k + 1$.

3 Consider $n = k + 1$. We want to show that $2^{k+1} > 2(k + 1) + 1$

$$\Rightarrow 2^{k+1} > 2k + 3$$

4 For $n = k + 1$,

$$2^{k+1} = 2.2^k$$
$$> 2(2k + 1)$$

Using the assumption.

$$= 2k + 2k + 2$$

We know that $2k + 2 > 3, \forall k \geq 3$ and so $2k + 2k + 2 > 2k + 3$

Hence $2^{k+1} > 2k + 3$ which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 3$, it is true $\forall n \geq 3, n \in \mathbb{N}$ by mathematical induction.

Induction can also be used to prove results from other spheres of mathematics such as calculus and matrices.

Example

Prove that $\frac{d}{dx}(x^n) = nx^{n-1}, \forall n \in \mathbb{Z}^+$.

1 For $n = 1$,

LHS $= \frac{d}{dx}(x^1) = 1$ RHS $= (1)x^{1-1} = x^0 = 1$

We know the LHS is equal to 1 as the gradient of $y = x$ is 1. However, we should prove this by differentiation by first principles as part of a proof.

Let $f(x) = x$

$$\frac{f(x+h) - f(x)}{h} = \frac{x + h - x}{h}$$
$$= \frac{h}{h}$$
$$= 1$$

Hence $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$, i.e.

$$\frac{d}{dx}(x^k) = kx^{k-1}$$

3 Consider $n = k + 1$. We want to show that $\frac{d}{dx}(x^{k+1}) = (k+1)x^k$.

4 For $n = k + 1$,

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x.x^k)$$
$$= 1.x^k + x.kx^{k-1}$$

Using the product rule and the assumption.

$$= x^k + kx^k$$
$$= x^k(1 + k)$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Example

Prove that $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \forall n \in \mathbb{Z}^+$ using mathematical induction.

1 For $n = 1$,

LHS $= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^1$ RHS $= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Since LHS = RHS, the result is true for $n = 1$.

2 Assume the result to be true for $n = k$,

i.e. $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$

3 Consider $n = k + 1$. We want to show that

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -(k+1) \\ 0 & 1 \end{pmatrix}.$$

4 For $n = k + 1$,

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{k+1}$$
$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^k$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

Using the assumption.

$$= \begin{pmatrix} 1+0 & -k-1 \\ 0+0 & 0+1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -(k+1) \\ 0 & 1 \end{pmatrix}$$

which is the required form.

5 So the result is true for $n = k + 1$ when true for $n = k$. Since the result is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Exercise 2

Prove these results using mathematical induction.

1 Prove that $2^{3n} - 1$ is divisible by 7, $\forall n \in \mathbb{Z}^+$.

2 Prove that $3^{2n} - 5$ is divisible by 4, $\forall n \in \mathbb{Z}^+$.

3 Prove that $5^n + 3$ is divisible by 4, $\forall n \in \mathbb{Z}^+$.

4 Prove that $9^n - 8n - 1$ is divisible by 64, $\forall n \in \mathbb{Z}^+$.

5 Prove that $6^n + 4$ is divisible by 10, $\forall n \in \mathbb{Z}^+$.

6 Prove that $n(n^2 - 1)(3n + 2)$ is divisible by 24, $\forall n \in \mathbb{Z}^+$.

7 Prove that $n! > 2^n$, for $n \geq 4, n \in \mathbb{Z}^+$.

8 Prove that $n! > n^2$, for $n \geq 4, n \in \mathbb{Z}^+$.

9 Prove that $2^n > n^3$, for $n \geq 10, n \in \mathbb{Z}^+$.

10 Find the smallest integer t for which $n! > 3^n$.
Hence prove by induction that $n! > 3^n$ for all $n > t$.

11 For $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, prove that $A^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix} \forall n \in \mathbb{Z}^+$.

12 Prove that $\frac{d^n}{dx^n}(e^{px}) = p^n e^{px}, \forall n \in \mathbb{Z}^+$.

13 Prove that $\frac{d^n}{dx^n}(\sin 2x) = 2^{n-1} \sin\left(2x + \frac{(n-1)\pi}{2}\right)$, for all positive integer values of n.

14 For $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & p & 0 \\ 0 & 0 & 3 \end{pmatrix}$, prove that $M^n = \begin{pmatrix} 1 & \frac{2(1-p^n)}{1-p} & 0 \\ 0 & p^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}, \forall n \in \mathbb{Z}^+$.

15 For $T = \begin{pmatrix} 4 & t \\ 0 & 1 \end{pmatrix}$, prove that $T^n = \begin{pmatrix} 4^n & \sum_{r=1}^{n} 4^{r-1} \\ 0 & 1 \end{pmatrix}, \forall n \in \mathbb{Z}^+$.

18.3 Forming and proving conjectures

For all of the examples met so far, the result to be proved was given in the question. This is not always the case; sometimes, it is necessary to form a conjecture which can then be proved using mathematical induction.

Example

Form a conjecture for the sum $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)}$.

In order to form the conjecture, consider the results for the first few values of n.

n	1	2	3	4	5
sum	$\frac{1}{2}$	$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$	$\frac{2}{3} + \frac{1}{12} = \frac{3}{4}$	$\frac{3}{4} + \frac{1}{20} = \frac{4}{5}$	$\frac{4}{5} + \frac{1}{30} = \frac{5}{6}$

Looking at the pattern, we can make a conjecture that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

We now try to prove this conjecture using mathematical induction.

This can be expressed as $\sum_{r=1}^{n} \frac{1}{r(r+1)}$.

1 For $n = 1$,

$$\text{LHS} = \sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2} \qquad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

Since LHS = RHS, the conjecture is true for $n = 1$.

2 Assume the result to be true for $n = k$,

i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

3 Consider $n = k + 1$. We want to show that

$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+1+1} = \frac{k+1}{k+2}$.

4 For $n = k + 1$,

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)}$$

$$= \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

Using the assumption.

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

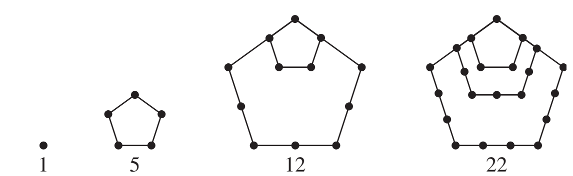
$$= \frac{k+1}{k+2}$$

which is the required form.

5 So the conjecture is true for $n = k + 1$ when true for $n = k$. Since it is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Example

Form a conjecture for the pentagonal numbers as shown below. Prove your conjecture by mathematical nduction.



So the sequence of pentagonal numbers begins 1, 5, 12, 22, 35, …

Remembering that these are formed by adding "a new layer" each time, we can consider this as a sum,

$1 + 4 + 7 + 10 + \ldots + (3n - 2)$

We are trying to find a formula for this. This is an arithmetic progression and so we can apply the formula for the sum to n terms with $a = 1$, $d = 3$.

Hence $S_n = \frac{n}{2}(2 + 3(n-1))$

$$= \frac{n}{2}(3n - 1)$$

$$= \frac{3n^2 - n}{2}$$

Again, our conjecture can be expressed using sigma notation:

$$\sum_{r=1}^{n} 3r - 2 = \frac{3n^2 - n}{2}$$

1 For $n = 1$,

$$\text{LHS} = \sum_{r=1}^{1} 3r - 2 = 3 - 2 = 1 \qquad \text{RHS} = \frac{3(1^2) - 1}{2} = \frac{3-1}{2} = 1$$

Since LHS = RHS, the conjecture is true for $n = 1$.

18 Mathematical Induction

2 Assume the result to be true for $n = k$,

i.e. $\sum_{r=1}^{k} 3r - 2 = \frac{3k^2 - k}{2}$

3 Consider $n = k + 1$. We want to show that

$$\sum_{r=1}^{k+1} 3r - 2 = \frac{3(k+1)^2 - (k+1)}{2}$$
$$= \frac{3k^2 + 6k + 3 - k - 1}{2}$$
$$= \frac{3k^2 + 5k + 2}{2}$$

4 For $n = k + 1$,

$$\sum_{r=1}^{k+1} 3r - 2$$
$$= \sum_{r=1}^{k} 3r - 2 + 3(k+1) - 2$$
$$= \frac{3k^2 - k}{2} + 3k + 1 \quad \text{Using the assumption.}$$
$$= \frac{3k^2 - k}{2} + \frac{6k + 2}{2}$$
$$= \frac{3k^2 + 5k + 2}{2}$$

which is the required form.

5 So the conjecture is true for $n = k + 1$ when true for $n = k$. Since it is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Example

For the matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, form a conjecture for $A^n, n \in \mathbb{Z}^+$. Prove your conjecture by mathematical induction.

To form the conjecture, find the results for the first few values of n.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$$
$$A^3 = \begin{pmatrix} 27 & 0 \\ 0 & 8 \end{pmatrix}$$
$$A^4 = \begin{pmatrix} 81 & 0 \\ 0 & 16 \end{pmatrix}$$

From this we can make a conjecture that $A^n = \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix}$.

We can now prove this using mathematical induction.

1 For $n = 1$,

$$\text{LHS} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{RHS} = \begin{pmatrix} 3^1 & 0 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Since LHS = RHS, the conjecture is true for $n = 1$.

2 Assume the result to be true for $n = k$,

i.e. $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 0 & 2^k \end{pmatrix}$

3 Consider $n = k + 1$. We want to show that $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^{k+1} & 0 \\ 0 & 2^{k+1} \end{pmatrix}$.

4 For $n = k + 1$,

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^k$$
$$= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 0 & 2^k \end{pmatrix}$$
$$= \begin{pmatrix} 3.3^k + 0 & 0 + 0 \\ 0 + 0 & 0 + 2.2^k \end{pmatrix} \quad \text{Using the assumption.}$$
$$= \begin{pmatrix} 3^{k+1} & 0 \\ 0 & 2^{k+1} \end{pmatrix}$$

which is the required form.

5 So the conjecture is true for $n = k + 1$ when true for $n = k$. Since the conjecture is true for $n = 1$, it is true $\forall n \in \mathbb{Z}^+$ by mathematical induction.

Exercise 3

1 For the matrix $D = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, form a conjecture for D^n. Prove your conjecture using mathematical induction.

2 Form a conjecture for the sum $5 + 8 + 11 + 14 + \ldots + (3n + 2)$. Prove your conjecture using mathematical induction.

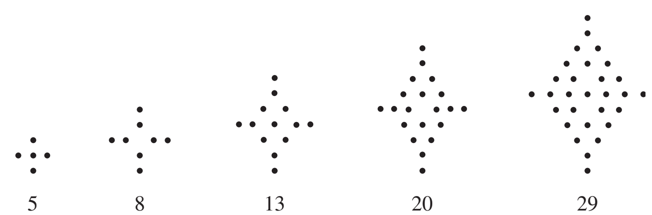
3 Form a conjecture for the series suggested by the initial values of $-3 + 1 + 5 + 9 + 13 + 17 + \ldots$. Prove your conjecture by mathematical induction.

4 With an unlimited supply of 4p and 7p stamps, make a conjecture about what values >17 of postage it is possible to create. Prove your conjecture using mathematical induction.

5 Make a conjecture about the sum of the first n odd numbers. Prove your conjecture using mathematical induction.

6 Find an expression for the nth term of the sequence $5, 10, 17, 26, 37, \ldots$.
Prove this result to be true using mathematical induction.

7 For the sequence below where each new pattern is made by adding a new "layer", make a conjecture for the number of dots in the nth term of the pattern. Prove this result to be true using mathematical induction.



Review exercise

1 Prove that $\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1), \forall n \in \mathbb{Z}^+$.

2 Prove that $n^5 - n$ is divisible by 5, $\forall\, n \in \mathbb{Z}^+$.

3 Prove that $\forall\, n \in \mathbb{Z}^+$, $11^{n+1} + 12^{2n-1}$ is divisible by 133.

4 Prove, using mathematical induction, that
$\frac{d^n}{dx^n}(xe^{px}) = p^{n-1}e^{px}(px+1), \ \forall\, n \in \mathbb{Z}^+$.

5 Prove, using mathematical induction, that for $T = \begin{pmatrix} 2 & 0 \\ p & 1 \end{pmatrix}$,
$T^n = \begin{pmatrix} 2^n & 0 \\ p(2^n - 1) & 1 \end{pmatrix}, \forall\, n \in \mathbb{Z}^+$.

6 Prove, using mathematical induction, that
$2 \cdot 6 \cdot 10 \cdot 14 \cdot \ldots \cdot (4n-2) = \frac{(2n)!}{n!}, \forall\, n \in \mathbb{Z}^+$.

7 Prove that $\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}, \forall\, n \in \mathbb{Z}^+$.

8 Form a conjecture for the sum $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n!$.
Prove your conjecture by mathematical induction.

9 Using mathematical induction, prove that $\frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$, for all positive integer values of n. [IB May 01 P2 Q4]

10 **a** Prove using mathematical induction that $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for all positive integer values of n.
b Determine whether or not this result is true for $n = -1$.
[IB May 02 P2 Q3]

11 Prove, using mathematical induction, that for a positive integer n,
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where $i^2 = -1$. [IB May 03 P2 Q3]

12 Using mathematical induction, prove that $\sum_{r=1}^{n}(r+1)2^{r-1} = n(2^n)$ for all positive integers. [IB May 05 P2 Q4]

13 The function f is defined by $f(x) = e^{px}(x+1)$, where $p \in \mathbb{R}$.
a Show that $f'(x) = e^{px}(p(x-1)+1)$.
b Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times. Use mathematical induction to prove that
$f^{(n)}(x) = p^{n-1}e^{px}(p(x+1)+n), n \in \mathbb{Z}^+$. [IB May 05 P2 Q2]

14 For $T = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & s \end{pmatrix}$, prove that $T^n = \begin{pmatrix} (-1)^n & 2^n - (-1)^n & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & s^n \end{pmatrix}, n \in \mathbb{Z}^+$
using mathematical induction. [IB May 06 P2 Q5]

15 Consider the sequence $\{a_n\}$, $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ where $a_1 = a_2 = 1$
and $a_{n+1} = a_n + a_{n-1}$ for all integers $n \geq 2$.
Given the matrix, $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ use the principle of mathematical induction
to prove that $Q^n = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix}$ for all integers $n \geq 2$.
[IB Nov 01 P2 Q4]

16 The matrix M is defined as $M = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.
a Find M^2, M^3 and M^4.
b **i** State a conjecture for M^n, i.e. express M^n in terms of n, where $n \in \mathbb{Z}^+$.
ii Prove this conjecture using mathematical induction.
[IB Nov 02 P2 Q1]