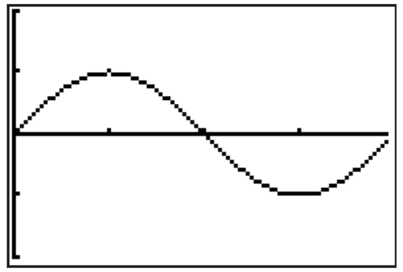
9 Differentiation 2 – Further Techniques

Leonhard Euler is considered to be one of the most important mathematicians of all time. He was born on 15 April 1707 in Basel, Switzerland, and died on 18 September 1783 in St Petersburg, Russia, although he spent much of his life in Berlin. Euler's mathematical discoveries are in many branches of mathematics including number theory, geometry, trigonometry, mechanics, calculus and analysis. Some of the best-known notation was created by Euler including the notation f(x) for a function, e for the base of natural logs, i for the square root of -1, π for pi, Σ for summation and many others. Euler enjoyed his work immensely, writing in 1741, "The King calls me his professor, and I think I am the happiest man in the world." Even on his dying day he continued to enjoy mathematics, giving a mathematics lesson to his grandchildren and doing some work on the motion of ballons.

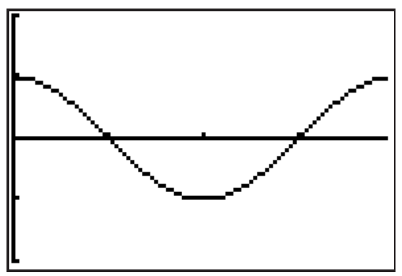
In Chapter 8, the basic concepts of differentiation were covered. However, the only functions that we differentiated all reduced to functions of the form $y = ax^n + \cdots + k$. In this chapter, we will meet and use further techniques to differentiate other functions. These include trigonometric, exponential and logarithmic functions, functions that are given implicitly, and functions that are the product or quotient of two (or more) functions.

9.1 Differentiating trigonometric functions

What is the derivative of sin x?



Using our knowledge of sketching the derived function, we know that the graph must be of this form:



We can use a calculator to draw the derivative graph as above.

This graph looks very much like the cosine function. We now need to see if it is.

In order to prove this, there are two results that need to be investigated. First, we need to consider what happens to $\dfrac{\sin h}{h}$ for small values of h. The calculator can be used to investigate this:

X	Y1
0	ERROR
.01	.99998
.02	.99993
.03	.99985
.04	.99973
.05	.99958
.06	.9994

X=0

It is clear that, as $h \to 0$, $\dfrac{\sin h}{h} \to 1$.

Second, we also need to investigate $\dfrac{\cos h - 1}{h}$ for small values of h.

X	Y1
0	ERROR
.01	.01745
.02	.01745
.03	.01745
.04	.01745
.05	.01745
.06	.01745

X=0

It is clear that, as $h \to 0$, $\dfrac{\cos h - 1}{h} \to 0$.

Now we can use differentiation by first principles to find the derivative of $f(x) = \sin x$.

$$f(x+h) = \sin(x+h)$$

$$\begin{aligned} f(x+h) - f(x) &= \sin(x+h) - \sin x \\ &= \sin x \cos h + \cos x \sin h - \sin x \\ &= \sin x(\cos h - 1) + \cos x \sin h \end{aligned}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h}$$

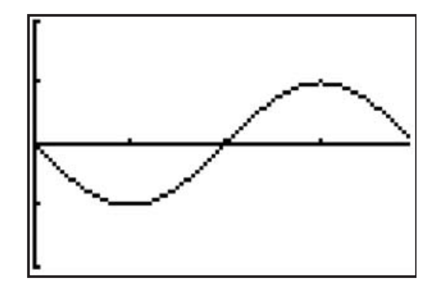
So $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = 0 + \cos x$ using the above results.

Hence $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \cos x$

Therefore if $f(x) = \sin x$, $f'(x) = \cos x$.

What about the derivative of cos x?

Examining the graph of the derived function using the calculator, this would appear to be $-\sin x$.



Again, we can use using differentiation by first principles to find the derivative of $f(x) = \cos x$.

$$f(x+h) = \cos(x+h)$$

$$\begin{aligned} f(x+h) - f(x) &= \cos(x+h) - \cos x \\ &= \cos x \cos h - \sin x \sin h - \cos x \\ &= \cos x(\cos h - 1) - \sin x \sin h \end{aligned}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

So $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = 0 - \sin x$ using the previous results.

Hence $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = -\sin x$

Therefore if $f(x) = \cos x$, $f'(x) = -\sin x$.

When dealing with trigonometric functions it is vital that radians are used. This is because of the results that we investigated above.

In degrees, $\lim\limits_{h \to 0} \dfrac{\sin h}{h} \neq 1$ as seen below.

X	Y1
0	ERROR
.01	.01745
.02	.01745
.03	.01745
.04	.01745
.05	.01745
.06	.01745

X=0

So in degrees, the derivative of sin x is not cos x.

Therefore, for calculus, we must always use radians.

Summarizing:

$y = \sin x$	$y = \cos x$
$\dfrac{dy}{dx} = \cos x$	$\dfrac{dy}{dx} = -\sin x$

It is clear that these are connected (as the two functions themselves are). Starting with $\sin x$, repeated differentiation gives $\cos x$, $-\sin x$, $-\cos x$ and then back to $\sin x$. This cycle can be remembered by

Differentiate ↓
S
C
−S
−C

We now have the derivatives of two of the six trigonometric functions. The other four functions are all defined in terms of $\sin x$ and $\cos x$ (remembering $\tan x = \dfrac{\sin x}{\cos x}$) and so this information provides the derivatives of the other four functions.

Proofs of these require the use of rules that have not yet been covered, and hence these are to be found later in the chapter. However, the results are shown below.

$y = \tan x$	$y = \csc x$	$y = \sec x$	$y = \cot x$
$\dfrac{dy}{dx} = \sec^2 x$	$\dfrac{dy}{dx} = -\csc x \cot x$	$\dfrac{dy}{dx} = \sec x \tan x$	$\dfrac{dy}{dx} = -\csc^2 x$

Example

Find the derivative of $y = \cos x - \sec x$.

$$\frac{dy}{dx} = -\sin x - \sec x \tan x$$

Example

Find the derivative of $y = 8 \sin x$.

$$\frac{dy}{dx} = 8 \cos x$$

Exercise 1

Find the derivative of each of these.

1 $y = \tan x + 3$
2 $y = \sin x - \csc x$
3 $y = \sin x + 6x^2$
4 $y = 5 \cos x$
5 $y = 7 \cot x$
6 $y = -3 \sec x$
7 $y = 9x^2 - 4 \cos x$
8 $y = 7x - 5 \sin x - \sec x$

9.2 Differentiating functions of functions (chain rule)

The chain rule is a very useful and important rule for differentiation. This allows us to differentiate composite functions. First consider $y = (ax + b)^n$.

Investigation

Consider these functions:

1 $y = (2x + 1)^2$
2 $y = (2x + 1)^3$
3 $y = (3x - 2)^4$
4 $y = (3x - 2)^5$
5 $y = (4 - x)^2$
6 $y = (4 - x)^3$

Using knowledge of the binomial theorem and differentiation, find the derivatives of the above functions. Factorise the answers.

You should have noticed a pattern that will allow us to take a "shortcut", which we always use, when differentiating this type of function.

This is that for functions of the form $y = (ax + b)^n$,

$$\frac{dy}{dx} = an(ax + b)^{n-1}$$

We can consider this as differentiating the bracket to the power *n* and then multiplying by the derivative of the bracket.

This is a specific case of a more general rule, known as the chain rule, which can be stated as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This is not due to cancelling!

This is where *y* is a function of a function. This means that we can consider *y* as a function of *u* and *u* as a function of *x*.

Proof

Consider $y = g(u)$ where $u = f(x)$.

If δx is a small increase in *x*, then we can consider δu and δy as the corresponding increases in *u* and *y*.

Then, as $\delta x \to 0$, δu and δy also tend to zero.

We know from Chapter 8 that $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0}\left(\dfrac{\delta y}{\delta x}\right)$

$$= \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}\right)$$

$$= \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u}\right) \cdot \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$$

$$= \lim_{\delta u \to 0}\left(\frac{\delta y}{\delta u}\right) \cdot \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$$

So $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

The use of this rule is made clear in the following examples.

Example

Differentiate $y = (3x - 4)^4$.

Let $u = 3x - 4$ and $y = u^4$.

Then $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 3$.

Hence $\frac{dy}{dx} = 4u^3 \cdot 3 = 12u^3$.

Substituting back for x gives $\frac{dy}{dx} = 12(3x - 4)^3$.

Example

Differentiate $y = (5 - 2x)^7$.

Let $u = 5 - 2x$ and $y = u^7$.

Then $\frac{dy}{du} = 7u^6$ and $\frac{du}{dx} = -2$.

Hence $\frac{dy}{dx} = 7u^6 \cdot -2 = -14u^6$.

Substituting back for x gives $\frac{dy}{dx} = -14(5 - 2x)^6$.

We will now apply the chain rule to other cases of a function of a function.

Example

Differentiate $y = \sin 4x$.

Let $u = 4x$ and $y = \sin u$.

Then $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 4$.

Hence $\frac{dy}{dx} = 4 \cos u$.

Substituting back for x gives $\frac{dy}{dx} = 4 \cos 4x$.

Example

Differentiate $y = \cos^2(3x)$.

Remember that this means $y = (\cos 3x)^2$.
Here there is more than one composition and so the chain rule must be extended to:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Let $v = 3x$ and $u = \cos v$ and $y = u^2$.

Then $\frac{dy}{du} = 2u$, $\frac{du}{dv} = -\sin v$ and $\frac{dv}{dx} = 3$.

Hence $\frac{dy}{dx} = -6u \sin v$.

Substituting back for x gives $\frac{dy}{dx} = -6 \cos 3x \sin 3x$.

This is the formal version of the working for chain rule problems. In practice, the substitution is often implied, as shown in the following examples. However, it is important to be able to use the formal substitution, both for more difficult chain rule examples and as a skill for further techniques in calculus.

Example

$f(x) = 2(7 - 3x)^6 - \tan 2x$

$f'(x) = 12(7 - 3x)^5 \cdot (-3) - 2\sec^2 2x$
$\quad = -36(7 - 3x)^5 - 2\sec^2 2x$

This working is sufficient and is what is usually done.

Example

$$f(x) = \tan(3x^2 - 4) + \frac{2}{\sqrt{2x - 1}} = \tan(3x^2 - 4) + 2(2x - 1)^{-\frac{1}{2}}$$

$$f'(x) = [\sec^2(3x^2 - 4) \cdot 6x] + \left[2 \cdot -\frac{1}{2}(2x - 1)^{-\frac{3}{2}} \cdot 2\right]$$
$$= 6x\sec^2(3x^2 - 4) - \frac{2}{(2x - 1)^{\frac{3}{2}}}$$

Exercise 2

Differentiate the following:

1 $f(x) = (x + 4)^2$ **2** $f(x) = (2x + 3)^2$ **3** $f(x) = (3x - 4)^2$

4 $f(x) = (5x - 2)^4$ **5** $f(x) = (5 - x)^3$ **6** $f(x) = (7 - 2x)^4$

7 $y = (9 - 4x)^5$ **8** $y = 4(2x + 3)^6$ **9** $y = (3x + 8)^{\frac{1}{2}}$

10 $y = (2x - 9)^{\frac{5}{3}}$ **11** $y = \sqrt[3]{6x - 5}$ **12** $y = \frac{1}{\sqrt{3x - 2}}$

13 $f(x) = \frac{4}{5x - 4}$ **14** $f(x) = \frac{7}{3 - 8x}$ **15** $P = \frac{3}{(4 - 3k)^2}$

16 $N = \frac{5}{\sqrt{(8 - 5p)^3}}$ **17** $y = \sin 4x$ **18** $y = \cos 3x$

19 $y = -\sin\frac{1}{2}x$ **20** $y = \tan 6x$ **21** $y = \sec 9x$

22 $y = 6x + \cot 3x$ **23** $y = \csc 2x + (3x + 2)^4$

24 $y = \sin 5x - \dfrac{4}{\sqrt{(3x + 4)^5}}$ **25** $y = \sin^3 x$ **26** $y = \tan^2(4x)$

27 $y = 3x^4 - \cos^3 x$ **28** $y = \dfrac{2}{(3x - 4)^5} - \sec^2(2x)$

29 $y = \cos\left(3x - \dfrac{\pi}{4}\right)$ **30** $y = \tan(\sqrt{x + 1})$

9.3 Differentiating exponential and logarithmic functions

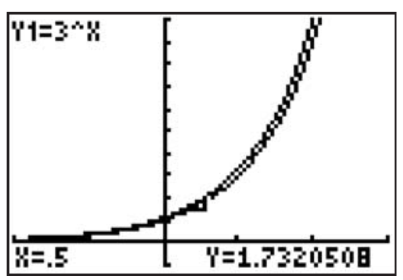
Investigation

Draw graphs of: (a) $y = 10^x$ (b) $y = 5^x$ (c) $y = 3^x$ (d) $y = 2^x$

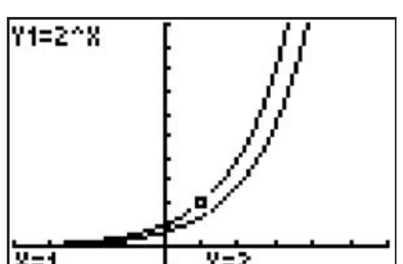
For each graph, draw the derivative graph, using a graphing calculator.

You should notice that the derivative graph is of a similar form to the original, that is, it is an exponential graph.

For $y = 3^x$, the derivative graph is just above the original.



For $y = 2^x$, the derivative graph is below the original.



This suggests that there is a function for which the derivative graph is identical to the original graph and that the base of this function lies between 2 and 3. What is this base?

This question was studied for many years by many mathematicians including Leonhard Euler, who first used the symbol e. The answer is that this base is e. Check that $y = e^x$ produces its own graph for the derived function on your calculator. Remember that $e = 2.71828\ldots$ is an irrational number.

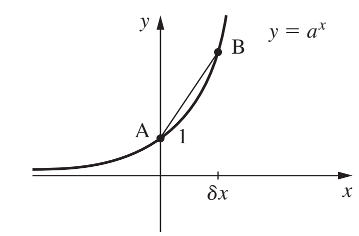
During the study of exponential functions in Chapter 5, we met the natural exponential function, $y = e^x$. The significance of this function becomes clearer now: the derivative of e^x is itself.

$$\frac{d}{dx}(e^x) = e^x$$

Below is a formal proof of this.

Proof of derivative of e^x

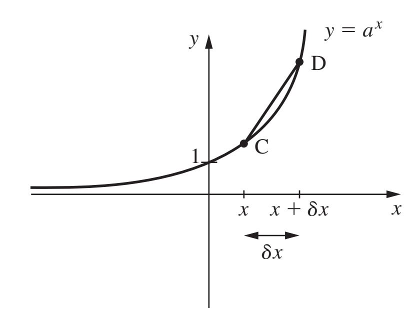
Consider the curve $y = a^x$.



Gradient of the chord AB $= \dfrac{a^{\delta x} - 1}{\delta x}$

Gradient at A $= \lim\limits_{\delta x \to 0}\left(\dfrac{a^{\delta x} - 1}{\delta x}\right)$

Now consider two general points on the exponential curve.



Gradient CD $= \dfrac{a^{x+\delta x} - a^x}{x + \delta x - x}$

Gradient at C $= \lim\limits_{\delta x \to 0} \dfrac{a^x(a^{\delta x} - 1)}{\delta x}$

Hence the gradient at C is a^x multiplied by the gradient at A.

But when $a \equiv e$, the gradient at $A = 1$ (this can be checked on a graphing calculator).
Gradient at $C = e^x \cdot 1$

Hence $\dfrac{d}{dx}(e^x) = e^x$

This is another property of the curve $y = e^x$. At (0,1) its gradient is 1.

Example

Differentiate $y = e^{5x}$.

This can be considered as $y = e^u$ where $u = 5x$.

Since $\frac{du}{dx} = 5$ and $\frac{dy}{du} = e^u$

$$\frac{dy}{dx} = e^u \cdot 5$$
$$= 5e^{5x}$$

Example

$y = e^{x^2 - 1}$

$$\Rightarrow \frac{dy}{dx} = 2xe^{x^2 - 1}$$

The result for e^x can be combined with the chain rule to create a general rule for differentiating exponential functions.

$$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$$

This now allows us to differentiate the inverse function of $y = e^x$, known as the **natural logarithmic function** $y = \ln x$.

As $y = e^x$ and $y = \ln x$ are inverse functions, from Chapter 5 we know that $e^{\ln x} = x$.

We can differentiate both sides of this equation (with respect to x).

$$\frac{d}{dx}(e^{\ln x}) = \frac{d}{dx}(x)$$
$$\Rightarrow e^{\ln x} \cdot \frac{d}{dx}(\ln x) = 1$$
$$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{e^{\ln x}}$$
$$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$$

This is a very important result.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

The result for $\ln x$ can be combined with the chain rule to create this general result:

If $\quad y = \ln(f(x))$

Then $\quad \frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

$$= \frac{f'(x)}{f(x)}$$

This result is particularly important for integration in Chapter 15.

Example

Differentiate $y = \ln(4x)$.

$$\frac{dy}{dx} = 4 \cdot \frac{1}{4x} = \frac{1}{x}$$

Example

Differentiate $y = \ln(\sin x)$.

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x$$
$$= \cot x$$

Example

Differentiate $y = \ln(3x - 2)$.

$$\frac{dy}{dx} = \frac{1}{3x - 2} \cdot 3$$
$$= \frac{3}{3x - 2}$$

Using the results for e^x and $\ln x$ helps us generalize so that we can find the derivatives of any exponential or logarithmic function.

To find out how to differentiate $y = a^x$ we first consider $y = 4^x$.

Since $e^{\ln 4} = 4$, we can rewrite this function as $y = (e^{\ln 4})^x$.

$$y = e^{x \ln 4}$$
$$\Rightarrow \frac{dy}{dx} = \ln 4 e^{x \ln 4}$$
$$\Rightarrow \frac{dy}{dx} = \ln 4 \cdot 4^x$$

In general, $a^x = (e^{\ln a})^x = e^{x \ln a}$

and so $\frac{dy}{dx} = \ln a \cdot e^{x \ln a} = \ln a \cdot a^x$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x$$

We will now look at $y = \log_a x$. In this case the change of base formula will help.

$$\log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x$$

Differentiating gives us

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$= \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Considering these two general results, it is clear that the results for e^x and $\ln x$ are actually just special cases.

Example

Differentiate $y = 3 \log_7 x$.

$$\frac{dy}{dx} = \frac{3}{x \ln 7}$$

Example

Differentiate $y = k \cdot 6^{2x}$.

$$\frac{dy}{dx} = 2k \ln 6 \cdot 6^{2x}$$

Sometimes it is useful to use laws of logarithms to assist the differentiation.

Example

Differentiate $y = \ln\left(\frac{2x + 1}{x - 4}\right)$.

We can consider this as $y = \ln(2x + 1) - \ln(x - 4)$.

So $\frac{dy}{dx} = \frac{2}{2x + 1} - \frac{1}{x - 4}$

$$= \frac{2(x - 4) - (2x + 1)}{(2x + 1)(x - 4)}$$

$$= \frac{-9}{(2x + 1)(x - 4)}$$

Exercise 3

Differentiate the following:

1 $f(x) = e^{3x}$ **2** $f(x) = e^{7x}$ **3** $f(x) = -e^{4x}$

4 $f(x) = \frac{2}{e^{5x}}$ **5** $f(x) = -\frac{6}{e^{9x}}$ **6** $f(x) = e^{x^2}$

7 $f(x) = e^{2x+3}$ **8** $f(x) = \ln 3x$ **9** $f(x) = \ln 7x$

10 $f(x) = -2 \ln 4x$ **11.** $f(x) = \ln(2x^2 + 4)$ **12** $y = 4^x$

13 $y = 10^x$ **14** $y = 6 \cdot 5^x$ **15** $y = e^{3x} - 3^x$

16 $y = \ln 2x - 2^x$ **17** $y = \log_2 x$ **18** $y = \log_8 x$

19 $y = 4^x - \log_5 x$ **20** $y = e^{4x} - \sin 2x + \ln x$ **21** $y = \ln(\cos x)$

22 $y = \ln(\tan x)$ **23** $y = \tan(\ln x)$

9.4 Product rule

Using the chain rule, $y = (2x + 3)^4$ can be differentiated without multiplying out the brackets first. However, this does not really help to differentiate $y = (3x + 2)(2x - 3)^3$ without some unpleasant simplification. Equally, we cannot currently differentiate $y = e^x \sin x$. These functions are products of two functions, and to be able to differentiate these we need to use the product rule.

For $y = uv$ where u and v are functions of x,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

This is sometimes remembered in the shortened form $\frac{dy}{dx} = v\,du + u\,dv$

Proof

Consider $y = uv$ where u and v are functions of x.

If δx is a small increase in x, and δu, δv and δy are the corresponding increases in u, v and y, then

$$y + \delta y = (u + \delta u)(v + \delta v) = uv + u\delta v + v\delta u + \delta u\delta v$$

As $y = uv$, $\delta y = u\delta v + v\delta u + \delta u\delta v$

So $$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x}$$

Now when $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\frac{\delta u}{\delta x} \to \frac{du}{dx}$, $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$, $\delta u \to 0$

Therefore $\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + 0$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example

Differentiate $y = (3x + 2)(2x - 3)^3$.

Let $y = uv$ where $u = 3x + 2$ and $v = (2x - 3)^3$.

$$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = 3(2x - 3)^2 \cdot 2$$
$$= 6(2x - 3)^2$$

$$\frac{dy}{dx} = (3x + 2) \cdot 6(2x - 3)^2 + 3(2x - 3)^3$$
$$= 3(2x - 3)^2[2(3x + 2) + 2x - 3]$$
$$= 3(2x - 3)^2(8x + 1)$$

There are often common factors which can be used to simplify the answer.

Example

Differentiate $y = e^x \sin x$.

Let $y = uv$ where $u = e^x$ and $v = \sin x$.

$$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = \cos x$$

This is the mechanics of the solution. It is not absolutely necessary for it to be shown as part of the solution.

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x$$
$$= e^x(\sin x + \cos x)$$

Example

Differentiate $y = 4x^2 \ln x$.

Let $y = uv$ where $u = 4x^2$ and $v = \ln x$.

$$\frac{du}{dx} = 8x \qquad \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = 8x \ln x + 4x^2 \cdot \frac{1}{x}$$
$$= 8x \ln x + 4x$$
$$= 4x(2 \ln x + 1)$$

Example

Differentiate $y = (2x + 1)^3 e^{2x} \cos x$.

This example is a product of three functions. We need to split it into two parts and then further split the second part.

Let $y = uv$ where $u = (2x + 1)^3$ and $v = e^{2x} \cos x$.

$$\frac{du}{dx} = 3(2x + 1)^2 \cdot 2$$

For $\frac{dv}{dx}$ we need to use the product rule again.

$$= 6(2x + 1)^2$$

Let $v = fg$ where $f = e^{2x}$ and $g = \cos x$

$$\frac{df}{dx} = 2e^{2x} \qquad \frac{dg}{dx} = -\sin x$$

$$\frac{dv}{dx} = 2e^{2x} \cos x - e^{2x} \sin x$$
$$= e^{2x}(2 \cos x - \sin x)$$

$$\frac{dy}{dx} = 6(2x + 1)^2 e^{2x} \cos x + (2x + 1)^3 e^{2x}(2 \cos x - \sin x)$$
$$= (2x + 1)^2 e^{2x}[6 \cos x + (2x + 1)(2 \cos x - \sin x)]$$

Exercise 4

Find the derivative of each of these.

1 $y = x^2 \sin x$ **2** $y = x^3 \cos x$ **3** $y = 3x^2 e^x$

4 $y = e^{3x} \sin x$ **5** $y = \ln x \sin x$ **6** $y = \sin x \cos x$

7 $y = \sin 3x \cos 2x$ **8** $y = x^2(x - 1)^2$ **9** $y = x^3(x - 2)^4$

10 $y = 2x^3(3x + 2)^2$ **11** $y = (x - 2)(2x + 1)^3$

12 $y = (x + 5)^2(3x - 2)^4$ **13** $y = (5 - 2x)^3(3x + 4)^2$

14 $y = (3x + 4)^3 \sin x$ **15** $y = 5^x \cos x$

16 $y = x^3 \log_6 x$ **17** $y = e^{4x} \sec 3x$

18 $y = (2x - 1)^3 \csc 3x$ **19** $y = 4^x \log_8 x$

20 $y = x \ln(2x + 3)$ **21** $y = 4x^2 \ln(x^2 + 2x + 5)$

22 $y = e^{3x} \sec\left(2x - \frac{\pi}{4}\right)$ **23** $y = \frac{3}{x^4} \tan\left(3x + \frac{\pi}{2}\right)$

24 $y = x^2 \ln x \sin x$ **25** $y = e^{3x}(x + 2)^2 \tan x$

9.5 Quotient rule

This rule is used for differentiating a quotient (one function divided by another) such as $y = \frac{u}{v}$.

Consider the function $y = \dfrac{(3x - 4)^2}{x^2} = x^{-2}(3x - 4)^2$

We can differentiate this using the product rule.

Let $y = uv$ where $u = (3x - 4)^2$ and $v = x^{-2}$.

$$\frac{du}{dx} = 6(3x - 4) \qquad \frac{dv}{dx} = -2x^{-3}$$

$$\frac{dy}{dx} = 6x^{-2}(3x - 4) - 2x^{-3}(3x - 4)^2$$

$= 2x^{-3}(3x-4)[3x-(3x-4)]$

$= 2x^{-3}(3x-4)(4)$

$= \dfrac{8(3x-4)}{x^3}$

In this case it was reasonably easy to rearrange into a product, but that is not always so. Generally, it is not wise to use the product rule for differentiating quotients as it often leads to an answer that is difficult to simplify, and hence a rule for differentiating quotients would be useful.

Consider $y = \dfrac{u}{v}$ where u and v are functions of x.

This can be written as $y = u \cdot \dfrac{1}{v}$.

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -v^{-2} \cdot \frac{d}{dx}(v)$$

$$= -\frac{1}{v^2} \cdot \frac{dv}{dx}$$

This is a slightly different application of the chain rule.

Using the product rule:

$$\frac{dy}{dx} = \frac{1}{v} \cdot \frac{du}{dx} + u \cdot -\frac{1}{v^2} \cdot \frac{dv}{dx}$$

$$= \frac{v}{v^2} \cdot \frac{du}{dx} + -\frac{u}{v^2} \cdot \frac{dv}{dx}$$

$$= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

This is the quotient rule:

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

This is often remembered as $\dfrac{dy}{dx} = \dfrac{v\,du - u\,dv}{v^2}$. The numerator of the quotient rule is very similar to the product rule but the sign is different.

Example

Differentiate $y = \dfrac{(3x-4)^2}{x^2}$ using the quotient rule.

Let $u = (3x-4)^2 \qquad v = x^2 \qquad v^2 = x^4$

$\dfrac{du}{dx} = 6(3x-4) \qquad \dfrac{dv}{dx} = 2x$

Once again, this is the mechanics of the solution and so does not necessarily need to be shown.

So $\dfrac{dy}{dx} = \dfrac{6x^2(3x-4) - 2x(3x-4)^2}{x^4}$

$= \dfrac{2x(3x-4)[3x-(3x-4)]}{x^4}$

$= \dfrac{2(3x-4)(4)}{x^3}$

$= \dfrac{8(3x-4)}{x^3}$

Example

Differentiate $y = \dfrac{e^{2x}}{\sin x}$.

Let $u = e^{2x} \qquad v = \sin x \qquad v^2 = \sin^2 x$

$\dfrac{du}{dx} = 2e^{2x} \qquad \dfrac{dv}{dx} = \cos x$

So $\dfrac{dy}{dx} = \dfrac{2e^{2x}\sin x - e^{2x}\cos x}{\sin^2 x}$

$= \dfrac{e^{2x}(2\sin x - \cos x)}{\sin^2 x}$

Example

Differentiate $y = \dfrac{\ln x}{(2x-5)^3}$.

Let $u = \ln x \qquad v = (2x-5)^3 \qquad v^2 = (2x-5)^6$

$\dfrac{du}{dx} = \dfrac{1}{x} \qquad \dfrac{dv}{dx} = 6(2x-5)^2$

So $\dfrac{dy}{dx} = \dfrac{\frac{1}{x}(2x-5)^3 - 6\ln x(2x-5)^2}{(2x-5)^6}$

$= \dfrac{\frac{1}{x}(2x-5) - 6\ln x}{(2x-5)^4}$

$= \dfrac{2x-5-6x\ln x}{x(2x-5)^4}$

Exercise 5

Use the quotient rule to differentiate these.

1 $f(x) = \dfrac{e^x}{\cos x}$ **2** $f(x) = \dfrac{6x^2}{x+3}$ **3** $f(x) = \dfrac{7x}{\tan x}$

4 $f(x) = \dfrac{\ln x}{4x}$ **5** $f(x) = \dfrac{e^x}{x-4}$ **6** $f(x) = \dfrac{x+3}{x-3}$

7 $f(x) = \dfrac{\sqrt{x+9}}{x^2}$ **8** $f(x) = \dfrac{4^x}{\sqrt{x}}$ **9** $f(x) = \dfrac{2x}{\sqrt{x-1}}$

10 $y = \dfrac{e^{3x}}{9x^2}$ **11** $y = \dfrac{\log_6 x}{x+6}$ **12** $y = \dfrac{\ln x}{\ln(x-4)}$

13 $y = \dfrac{e^x}{e^x - e^{-x}}$ **14** $y = \dfrac{\sin 2x}{e^{6x}}$ **15** $y = \dfrac{4(3x-2)^5}{(2x+3)^3}$

16 $y = \dfrac{x \sin x}{e^x}$ **17** $y = \dfrac{x^2 e^{3x}}{(x+5)^2}$ **18** $y = \dfrac{\sec\left(x + \frac{\pi}{4}\right)}{e^{2x}}$

19 $y = \dfrac{\cot\left(2x - \frac{\pi}{3}\right)}{\ln(3x+1)}$

20 Use the quotient rule to prove the results for tan x, csc x, sec x and cot x. You need to remember that $\tan x = \dfrac{\sin x}{\cos x}$.

Also consider how you could have proved these two results using only the chain rule.

9.6 Implicit differentiation

This is the differentiation of functions that are stated implicitly. Until now we have mostly considered functions that are stated explicitly, that is, $y = \ldots$

Functions defined implicitly have equations that are not in the form $y = \ldots$ Some of these equations are easily made explicit (such as $2x + 3y = 5$) but others are more difficult to rearrange. Some of these implicit equations may be familiar, such as the circle equation $(x-4)^2 + (y+3)^2 = 36$. Differentiating implicit functions does not require any further mathematical techniques than those covered so far. The key concept utilized in implicit differentiation is the chain rule.

Method for implicit differentiation

1. Differentiate each term, applying the chain rule to functions of the variable.
2. Rearrange the answer to the form $\dfrac{dy}{dx}$.

Example

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $3x^2 + y^2 = 7$

It is possible to rearrange this function to an explicit form. However, unless you are told otherwise, it is often better to leave it in this form and differentiate implicitly.

Differentiating with respect to x:

$$6x + 2y\frac{dy}{dx} = 0$$

Applying the chain rule gives $\dfrac{d}{dx}(y^2) = 2y \cdot \dfrac{dy}{dx}$.

$$\Rightarrow 2y\frac{dy}{dx} = -6x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3x}{y}$$

Note that the answer contains both x and y.

We can now find the second derivative by differentiating this again.

Using the quotient rule:

In a case like this it is important to be able to explicitly state $\dfrac{dy}{dx} = \ldots$ so that the second derivative can be found, but this is not always the situation.

$$\frac{d^2y}{dx^2} = \frac{-3y + 3x\dfrac{dy}{dx}}{y^2}$$

$$= \frac{-3y + 3x\left(-\dfrac{3x}{y}\right)}{y^2}$$

$$= \frac{-3y - \dfrac{9x^2}{y}}{y^2}$$

$$= \frac{-3y^2 - 9x^2}{y^3}$$

We would usually leave the answer in this form. However, if we wanted $\dfrac{d^2y}{dx^2}$ as a function of x, we could proceed as follows:

We know that $3x^2 + y^2 = 7$ $\Rightarrow y^2 = 7 - 3x^2$

$$\frac{d^2y}{dx^2} = \frac{-3(7-3x^2) - 9x^2}{(7-3x^2)^{\frac{3}{2}}}$$

$$= \frac{-21}{(7-3x^2)^{\frac{3}{2}}}$$

Example

Find $\dfrac{dy}{dx}$ for $6 \sin x - e^{3x}y^3 = 9$.

Differentiating with respect to x:

Use the product rule to differentiate $e^{3x}y^3$.

$$6 \cos x - \left(3e^{3x}y^3 + e^{3x}3y^2\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 3e^{3x}y^2\left(y + \frac{dy}{dx}\right) = 6 \cos x$$

$$\Rightarrow y + \frac{dy}{dx} = \frac{2 \cos x}{e^{3x}y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 \cos x}{e^{3x}y^2} - y$$

Example

Find $\frac{dQ}{dp}$ for $\sin \pi p + \frac{2Q}{(p+3)^2} = \ln p$.

Differentiating with respect to p:

$$\pi \cos \pi p + \frac{2\frac{dQ}{dp}(p+3)^2 - 2(p+3)\cdot 2Q}{(p+3)^4} = \frac{1}{p}$$

$$\Rightarrow \frac{2\frac{dQ}{dp}(p+3)^2 - 4Q(p+3)}{(p+3)^4} = \frac{1}{p} - \pi \cos \pi p$$

$$\Rightarrow \frac{2\frac{dQ}{dp}}{(p+3)^2} - \frac{4Q}{(p+3)^3} = \frac{1}{p} - \pi \cos \pi p$$

$$\Rightarrow \frac{2\frac{dQ}{dp}}{(p+3)^2} = \frac{1}{p} - \pi \cos \pi p + \frac{4Q}{(p+3)^3}$$

$$\Rightarrow 2\frac{dQ}{dp} = \frac{(p+3)^2}{p} - \pi(p+3)^2 \cos \pi p + \frac{4Q}{p+3}$$

$$\Rightarrow \frac{dQ}{dp} = \frac{(p+3)^2}{2p} - \frac{\pi}{2}(p+3)^2 \cos \pi p + \frac{2Q}{p+3}$$

Example

Find the equations of the tangents to $3x^2y - y^2 = 27$ when $x = 2$.

Differentiating with respect to x:

$$6xy + 3x^2\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(3x^2 - 2y) = -6xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6xy}{3x^2 - 2y}$$

To find $\frac{dy}{dx}$ we now require the y-coordinates. So, from the formula

$3x^2y - y^2 = 27$, when $x = 2$, we find

$$12y - y^2 = 27$$
$$\Rightarrow y^2 - 12y + 27 = 0$$
$$\Rightarrow (y-3)(y-9) = 0$$
$$\Rightarrow y = 3, y = 9$$

At (2, 3)

$$\frac{dy}{dx} = \frac{-6\cdot 2\cdot 3}{12 - 6} = \frac{-36}{6} = -6$$

So the equation of the tangent is

$$y - 3 = -6(x - 2)$$
$$\Rightarrow y = -6x + 15$$

At (2, 9)

$$\frac{dy}{dx} = \frac{-6\cdot 2\cdot 9}{12 - 18} = \frac{-108}{-6} = 18$$

So the equation of the tangent is

$$y - 9 = 18(x - 2)$$
$$\Rightarrow y = 18x - 27$$

Some questions will require the second derivative to be found, and a result to be shown to be true that involves $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y. In the examples so far we have found $\frac{dy}{dx} = \ldots$ and then differentiated this again with respect to x to find $\frac{d^2y}{dx^2}$. With other questions, it is best to leave the result as an implicit function and differentiate for a second time, implicitly. The following two examples demonstrate this.

Example

Show that $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = -\frac{1}{x}$ for $x^2y + x \ln x = 6x$.

Differentiating with respect to x:

$$2xy + x^2\frac{dy}{dx} + \ln x + \frac{1}{x}\cdot x = 6$$

$$\Rightarrow 2xy + x^2\frac{dy}{dx} + \ln x + 1 = 6$$

Differentiating again with respect to x:

$$2y + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + \frac{1}{x} = 0$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = -\frac{1}{x}$$

Example

Given that $e^xy = \cos x$, show that $2y + 2\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$.

Differentiating with respect to x:

$$e^xy + e^x\frac{dy}{dx} = -\sin x$$

Differentiating again with respect to x:

$$e^xy + e^x\frac{dy}{dx} + e^x\frac{dy}{dx} + e^x\frac{d^2y}{dx^2} = -\cos x$$

From the original function, $-\cos x = -e^xy$

So we have $e^xy + e^x\frac{dy}{dx} + e^x\frac{dy}{dx} + e^x\frac{d^2y}{dx^2} = -e^xy$

$$\Rightarrow e^x y + 2e^x\frac{dy}{dx} + e^x\frac{d^2y}{dx^2} = -e^x y$$

$$\Rightarrow 2e^x y + 2e^x\frac{dy}{dx} + e^x\frac{d^2y}{dx^2} = 0$$

Dividing by e^x (since $e^x \neq 0\ \forall x \in \mathbb{R}$):

$$\Rightarrow 2y + 2\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$$

Exercise 6

1 Find $\frac{dy}{dx}$ for:

a $x^3 + xy = 4$ **b** $4x^2 + y^2 = 9$ **c** $y^3 - \sqrt{x} = 0$

d $(x + 3)(y + 2) = \ln x$ **e** $xy = y^2 - 7$ **f** $e^y = (x - y)^2$

g $e^{2x}y^3 = 9 - \sin 3x$ **h** $y = \cos(x + y)$ **i** $x^4 = y \ln y$

j $(x + y)^3 = e^y$ **k** $\frac{(x + y)^4}{y} = 8x - e^x$

2 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for:

a $4y + 3y^2 = x^2$ **b** $4xy = \sin x - y$ **c** $xe^y = 8$

3 For the function defined implicitly by $x^4 - 2xy + y^2 = 4$, find the equations of the tangents at $x = 1$.

4 Show that $2y + \frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$ for $e^x y = \sin x$.

5 Given that $xy = \sin x$, show that $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + x^2y = 0$.

6 Show that $x^3\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + xy = -2$ for $xy = \ln x$.

7 Given that $e^{2x}y = \ln x$, show that $x^2e^{2x}\left(4y + 4\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = -1$.

9.7 Differentiating inverse trigonometric functions

In order to find the derivative of $\sin^{-1} x$ (or arcsin x), we apply implicit differentiation.

Consider $y = \sin^{-1} x$

$\Rightarrow x = \sin y$

Differentiating with respect to x:

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

This is because $\sin y = x \Rightarrow \sin^2 y = x^2$

$\Rightarrow 1 - \cos^2 y = x^2$

$\Rightarrow \cos^2 y = 1 - x^2$

$\Rightarrow \cos y = \sqrt{1 - x^2}$

For $y = \sin^{-1} x$ $\quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

We can now consider $y = \sin^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow \frac{x}{a} = \sin y$$

$$\Rightarrow x = a \sin y$$

Differentiating with respect to x:

$$1 = a \cos y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a \cos y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}}$$

$$= \frac{\sqrt{a^2}}{a\sqrt{a^2 - x^2}}$$

$$= \frac{1}{\sqrt{a^2 - x^2}}$$

This is because $\sin y = \frac{x}{a} \Rightarrow 1 - \cos^2 y = \frac{x^2}{a^2}$

$\Rightarrow 1 - \cos^2 y = \frac{x^2}{a^2}$

For $y = \sin^{-1}\left(\frac{x}{a}\right)$ $\quad \frac{d}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$

Similarly $\frac{dy}{dx}$ can be obtained for $y = \cos^{-1}(x)$ and $y = \cos^{-1}\left(\frac{x}{a}\right)$.

For $y = \cos^{-1} x$ $\quad \frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$

For $y = \cos^{-1}\left(\frac{x}{a}\right)$ $\quad \frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}}$

Now consider $y = \tan^{-1}(x)$

$\Rightarrow x = \tan y$

Differentiating with respect to x:

$$1 = \sec^2 y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2 + 1}$$

Remember that $\sec^2 x = \tan^2 x + 1$

So $\sec^2 y = \tan^2 y + 1$
$= x^2 + 1$

For $y = \tan^{-1}x$ $\quad \dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$

A similar result can be found for $\tan^{-1}\left(\dfrac{x}{a}\right)$.

For $y = \tan^{-1}\left(\dfrac{x}{a}\right)$ $\quad \dfrac{dy}{dx} = \dfrac{a}{a^2 + x^2}$

Example

Differentiate $y = \cos^{-1}\left(\dfrac{x}{3}\right)$.

$$\frac{dy}{dx} = \frac{-1}{\sqrt{9 - x^2}}$$

Example

Differentiate $y = \sin^{-1}(4x)$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{1}{16} - x^2}}$$

$$= \frac{1}{\sqrt{\frac{1}{16}}\sqrt{1 - 16x^2}}$$

$$= \frac{4}{\sqrt{1 - 16x^2}}$$

In this case $a = \dfrac{1}{4}$.

We could also consider these examples to be applications of the chain rule. This may be easier and shorter (but both methods are perfectly valid). This is demonstrated below.

$$y = \sin^{-1}(4x)$$

Then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - (4x)^2}} \cdot 4$

$$= \frac{4}{\sqrt{1 - 16x^2}}$$

In some cases it is not possible to use the stated results, and the chain rule must be applied.

Example

Differentiate $y = \tan^{-1}\sqrt{x}$.

Here we must use the chain rule.

$$\frac{dy}{dx} = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}(1 + x)}$$

Exercise 7

Differentiate the following functions.

1 $y = \sin^{-1}\left(\dfrac{x}{5}\right)$

2 $y = \cos^{-1}\left(\dfrac{x}{8}\right)$

3 $y = \tan^{-1}\left(\dfrac{x}{10}\right)$

4 $y = \sin^{-1}\left(\dfrac{2x}{3}\right)$

5 $y = \cos^{-1}(3x)$

6 $y = \tan^{-1}\left(\dfrac{e^x}{2}\right)$

7 $y = \cos^{-1}\sqrt{x + 4}$

8 $y = \tan^{-1}(2x - 1)$

9 $y = \sin^{-1}(\ln 5x)$

9.8 Summary of standard results

This chapter has covered a variety of techniques including the chain rule, product rule, quotient rule and implicit differentiation. These have produced a number of standard results, which are summarized below.

$y =$	$\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x \ln a}$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$

Within this chapter and Chapter 8, we have covered all of the differentiation techniques and skills for IB Higher Level. One of the key skills in an examination is to be able to identify which technique is required to solve a particular problem. Exercise 8 contains a mixture of examples that require the knowledge and use of standard results and the above techniques.

Exercise 8

Differentiate the following functions using the appropriate techniques and results.

1 $f(x) = x^2 - 5x + 9$ **2** $y = (2x - 7)^3$ **3** $f(x) = \cos 8x - \sqrt{9x}$

4 $y = \sec x - e^{5x}$ **5** $f(x) = x^3e^{-4x}$ **6** $y = x^2 \ln x$

7 $f(x) = \frac{\sin 3x}{e^x}$ **8** $y = \frac{x \sin x}{e^{4x}}$ **9** $f(x) = 3^x \sin x$

10 $y = \frac{\log_2 x}{(x-4)^3}$ **11** $f(x) = \frac{x^2 \ln x}{x+9}$ **12** $y = 3 \cos 2x \sin 4x$

13 $y = 6 \sin^{-1} 2x$ **14** $f(x) = \frac{\cos^{-1} x}{3x^2}$ **15** $y = x \sin x \ln x$

16 $y = \frac{\ln(\cot x)}{e^x}$ **17** Find $f'(2)$ for $f(x) = \frac{x^3(x-7)^2}{(2x-1)^3}$.

18 Find $f'(4)$ for $f(x) = \frac{1}{x}\tan^{-1}\left(\frac{x}{4}\right)$. **19** Find $\frac{dy}{dx}$ for $x^2y + e^xy^2 = 9$.

20 Find $\frac{dy}{dx}$ for $x^4y^3 - y \sin x = 2$.

9.9 Further differentiation problems

The techniques covered in this chapter can also be combined to solve differentiation problems of various types, including equations of tangents and normals, and stationary points. Problems of this type are given in Exercise 9.

Example

Find the stationary point for $\frac{y}{e^x} = x$ and determine its nature.

In this case, it is easiest to consider this as $y = xe^x$.

Using the product rule, $\frac{dy}{dx} = e^x x + e^x$
$= e^x(x + 1)$

For stationary points, $\frac{dy}{dx} = 0$

Hence $e^x(x + 1) = 0$
$\Rightarrow x = -1$

At $x = -1$,
$y = e^{-1} \cdot -1$
$= -\frac{1}{e}$

Hence the stationary point is $\left(-1, -\frac{1}{e}\right)$

$\frac{d^2y}{dx^2} = e^x(x + 1) + e^x$
$= e^x(x + 2)$

So, at $x = -1$, $\frac{d^2y}{dx^2} = e^{-1}(1)$
$= \frac{1}{e}$

So $\frac{d^2y}{dx^2} > 0$, therefore $\left(-1, -\frac{1}{e}\right)$ is a local minimum turning point.

Exercise 9

1 Find the gradient of the tangent to $y = \tan^{-1} 3x$ where $x = \frac{1}{\sqrt{3}}$.

2 Find the gradient of the tangent to $y = \ln\sqrt{1 - \cos 2x}$ where $x = \frac{\pi}{4}$.

3 Given $y = \frac{4^x}{e^x(x + 2)}$, find the rate of change where $x = 2$.

4 Find the equation of the tangents to $x^2y + y^2 = 6$ at $x = 1$.

5 Find the gradient of the tangent to $2x \ln x - y \ln y = 2e(1 - e)$ at the point (e, e^2).

6 Find the value of $\frac{d^2y}{dx^2}$ when $x = \pi$ for $\frac{y}{x} = \frac{\sin 2x}{e^x}$.

7 Find the stationary points of $y = \frac{x^2}{e^x}$.

8 Find the stationary points of $y = 4x^2 \ln x, x > 0$.

9 Find the stationary points, and their nature, of the curve given by $y = \frac{x^3}{e^x}$.

10 Show that the gradient of the tangent to the curve given by
$\frac{xy}{\pi} + \sin x \ln x = \cos x + 1$ at $x = \pi$ is $\ln \pi$.

Review exercise

1 Differentiate these functions.
a $y = 5(3x - 2)^4$ **b** $f(x) = \frac{7}{\sqrt{(3 - 2x^2)}}$ **c** $y = 6t - \sec 3t$
d $f(x) = 6e^{8x}$ **e** $y = \ln 6x - 3^x$

2 Differentiate these functions.
a $y = e^{4x} \sin 3x$ **b** $y = \ln(x \sin x)$ **c** $f(x) = \frac{e^{5x}}{\sqrt{x + 4}}$
d $y = \ln\left(\frac{3x + 4}{2x - 1}\right)$ **e** $y = \log_{10}\left(\frac{e^{2x} \cos 3x}{(x + 4)^2}\right)$

3 Find $\frac{dy}{dx}$ for: **a** $4y^2 - 3x^2y = 5$ **b** $x^3 = y \ln x$

4 Find $\frac{d^2y}{dx^2}$ for $x^2 \sin x - e^x y = 7$.

5 Differentiate $y = 2 \tan^{-1}\left(\frac{1 + \cos x}{\sin x}\right)$.

6 Find the exact value of the gradient of the tangent to $y = \frac{1}{x \sin x}$ where $x = \frac{\pi}{4}$.

7 Find the gradient of the tangent to $3x^2 + 4y^2 = 7$ at the point where $x = 1$ and $y > 0$. [IB May 01 P1 Q4]

8 A curve has equation $xy^3 + 2x^2y = 3$.
Find the equation of the tangent to this curve at the point (1, 1). [IB May 02 P1 Q17]

9 A curve has equation $x^3y^2 = 8$.
Find the equation of the normal to the curve at the point (2,1). [IB May 03 P1 Q10]

10 Find the stationary points of $y = x^2 \tan^{-1} x$.

11 Find the stationary points of $y = \frac{e^{2x} \sin x}{x + 1}$ for $0 < x < \pi$.

12 Show that the point $P(2, -2)$ lies on the circle with equation $(x + 2)^2 + (y - 2)^2 = 32$ and the parabola with equation $y^2 = 12 - 4x$. Also show that these curves share a common tangent at P, and state the equation of this tangent.

13 If $y = \ln(2x - 1)$, find $\frac{d^2y}{dx^2}$. [IB Nov 04 P1 Q5]

14 Consider the function $f(t) = 3 \sec 2t + 5t$.
a Find $f'(t)$.
b Find the **exact** values of
i $f(\pi)$
ii $f'(\pi)$. [IB Nov 03 P1 Q8]

15 Consider the equation $2xy^2 = x^2y + 3$.
a Find y when $x = 1$ and $y < 0$.
b Find $\frac{dy}{dx}$ when $x = 1$ and $y < 0$. [IB Nov 03 P1 Q15]