5 Exponential and Logarithmic Functions

In this chapter we will meet logarithms, which have many important applications, particularly in the field of natural science. Logarithms were invented by John Napier as an aid to computation in the 16th century.


John Napier

John Napier was born in Edinburgh, Scotland, in 1550. Few records exist about John Napier's early life, but it is known that he was educated at St Andrews University, beginning in 1563 at the age of 13. However, it appears that he did not graduate from the university as his name does not appear on any subsequent pass lists. The assumption is that Napier left to study in Europe. There is no record of where he went, but the University of Paris is likely, and there are also indications that he spent time in Italy and the Netherlands.

While at St. Andrews University, Napier became very interested in theology and he took part in the religious controversies of the time. He was a devout Protestant, and his most important work, the *Plaine Discovery of the Whole Revelation of St. John* was published in 1593.

It is not clear where Napier learned mathematics, but it remained a hobby of his, with him saying that he often found it hard to find the time to work on it alongside his work on theology. He is best remembered for his invention of logarithms, which were used by Kepler, whose work was the basis for Newton's theory of gravitation. However his mathematics went beyond this and he also worked on exponential expressions for trigonometric functions, the decimal notation for fractions, a mnemonic for formulae used in solving spherical triangles, and "Napier's analogies", two formulae used in solving spherical triangles. He was also the inventor of "Napier's bones", used for mechanically multiplying, dividing and taking square and cube roots. Napier also found exponential epressions for trignometric functions, and introduced the decimal notation for fractions.

We can still sympathize with his sentiments today, when in the preface to the *Mirifici logarithmorum canonis descriptio*, Napier says he hopes that his "logarithms will save calculators much time and free them from the *slippery errors* of calculations".

5.1 Exponential functions

An exponential (or power) function is of the form $y = a^x$.

a is known as the base $(a \neq 1)$.

x is known as the exponent, power or index.

Remember the following rules for indices:

1. $a^p \times a^q = a^{p+q}$
2. $\dfrac{a^p}{a^q} = a^{p-q}$
3. $(a^p)^q = a^{pq}$
4. $\dfrac{1}{a^p} = a^{-p}$
5. $\sqrt[q]{a^p} = a^{\frac{p}{q}}$
6. $a^0 = 1$

Example

Simplify $\dfrac{x^{\frac{1}{5}} \times x^{\frac{2}{5}}}{\sqrt[5]{x^3}}$.

$$\frac{x^{\frac{1}{5}} \times x^{\frac{2}{5}}}{\sqrt[5]{x^3}} = \frac{x^{\frac{3}{5}}}{x^{\frac{3}{5}}} = x^0 = 1$$

Example

Evaluate $8^{-\frac{2}{3}}$ without a calculator.

$$8^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{2^2} = \frac{1}{4}$$

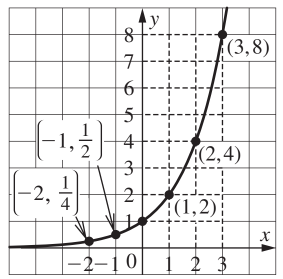
Graphing exponential functions

Consider the function $y = 2^x$.

x	-2	-1	0	1	2	3	4	5
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32

The y-values double for every integral increase of x.

The first few points are shown in this graph:



Investigation

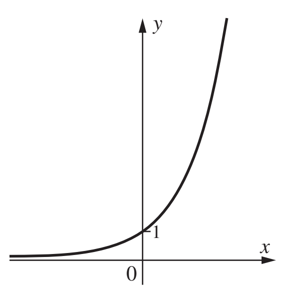
Using a graphing calculator, sketch these graphs:

(a) $y = 3^x$ (b) $y = 4^x$ (c) $y = 5^x$ (d) $y = 10^x$

Try to identify a pattern.

The investigation should have revealed that all exponential graphs

1. pass through the point (0,1)
2. have a similar shape
3. are entirely above the x-axis.

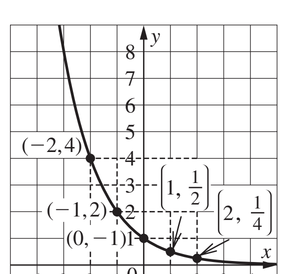


This shape is known as exponential growth, and all graphs of the form $y = a^x, a > 1$ have this shape. The domain restriction of $a > 1$ is important. We know that when $a = 1$ the graph is the horizontal line $y = 1$, and below we will see what happens when $0 < a < 1$.

For graphs of the form $y = a^x, 0 < a < 1$ let us consider $y = \left(\frac{1}{2}\right)^x$.

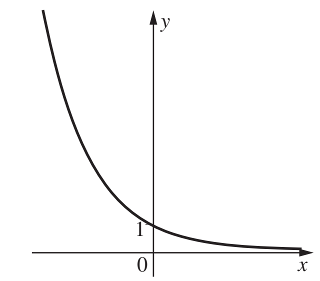
x	−2	−1	0	1	2	3	4	5
y	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$

The first few points are shown in this graph:



Now $y = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$. All exponential graphs of the form $y = a^x, 0 < a < 1$ can be expressed in this way, and from our knowledge of transformations of functions this is actually a reflection in the y-axis.

Hence this is the general graph of $y = a^{-x}, a > 1$:



This is known as exponential decay. Exponential decay graphs can be expressed as $y = a^{-x}, a > 1$ or as $y = a^x, 0 < a < 1$ as shown above.

Exercise 1

1 Simplify these.

a $p^4 \times p^5$ **b** $\frac{p^7}{p^2}$ **c** $(x^3)^5$ **d** $3y^2 \times 7y^3$

e $(2x^3)^4$ **f** $t^4 \times t^{-2}$ **g** $\frac{8p^6}{4p^4}$ **h** $\frac{18p^5}{3p^{-2}}$

2 Without using a calculator, evaluate these.

a $16^{\frac{1}{2}}$ **b** $81^{\frac{1}{4}}$ **c** 10^{-1}

d 19^0 **e** $25^{\frac{3}{2}}$ **f** $9^{-\frac{1}{2}}$

g $8^{-\frac{2}{3}}$ **h** $4^{-\frac{3}{2}}$ **i** $\left(\frac{1}{27}\right)^{-\frac{2}{3}}$

3 Simplify these.

a $\frac{x^5 \times x^3}{x^2}$ **b** $\frac{4y^3 \times 2y^6}{6y^5}$

c $\frac{5p^3 \times 2p^{-5}}{p^4}$ **d** $\frac{t^{\frac{1}{2}} \times t^3}{t^{\frac{3}{2}}}$

e $\frac{4m^{\frac{5}{3}} \times 3m^{-\frac{5}{3}}}{2m^{\frac{2}{3}}}$ **f** $3x^2(4x^3 + 5x^{-1})$

g $x^{\frac{1}{2}}(2x^{\frac{1}{2}} + x^{-\frac{1}{2}})$ **h** $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2$

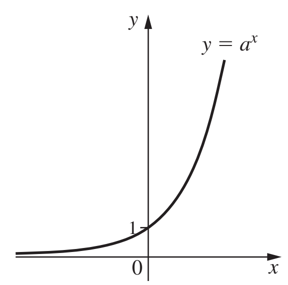
4 Draw the graph of each of these.

a $y = 3^x$ **b** $y = 5^x$ **c** $y = 6^x$ **d** $y = 10^x$

e $y = \left(\frac{1}{4}\right)^x$ **f** $y = 6^{-x}$ **g** $y = \left(\frac{3}{2}\right)^x$ **h** $y = \left(\frac{2}{3}\right)^x$

5.2 Logarithmic graphs

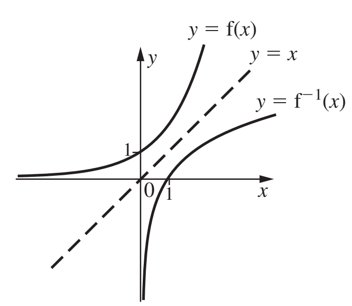
In the study of inverse functions, it was found that an inverse function exists only for one-to-one functions. The question is whether there is an inverse function for exponential functions.



Using the tests of horizontal and vertical lines, it is clear that any of these lines pass through only one point (or no point) on the graph.

So, for any exponential function $y = a^x (a \neq 1)$, an inverse function exists.

In Chapter 3, we also found that the graph of an inverse function is the reflection of the original function in the line $y = x$. Using this, we can find the shape of the inverse function.



As all exponential graphs have the same shape, all inverse graphs will also have the same shape. These inverse functions are known as **logarithmic functions**.

Logarithmic functions are defined

$$y = a^x \Leftrightarrow x = \log_a y$$

Consider the exponential function $y = 2^x$.

$y = 2^x \Rightarrow x = \log_2 y$

This means that the inverse function is written $y = \log_2 x$.

In $\log_2 x$, 2 is known as the base.

There are two key features of logarithmic graphs:

1. Logarithmic functions are defined only for $x > 0$.
2. All logarithmic graphs pass through (1,0).

This means that we can summarize the domain and range for exponential and logarithmic functions.

	Domain	Range
Exponential	$\mathbb{R}$	$y > 0$
Logarithmic	$x > 0$	$\mathbb{R}$

Interpreting a logarithm

A logarithm can be interpreted by "the answer to a logarithm is a power".

This comes from the definition:

$$\log_a q = p \Leftrightarrow a^p = q$$

So, for example, $\log_2 64 = x \Leftrightarrow 2^x = 64 \Rightarrow x = 6$.

Example

Find $\log_5 125$.

This is asking "What power of 5 gives 125?"

$\log_5 125 = x \Rightarrow 5^x = 125$

$\Rightarrow x = 3$

Example

Evaluate (a) $\log_{25} 5$ (b) $\log_5\left(\frac{1}{25}\right)$

(a) $\log_{25} 5 = x$

$\Rightarrow 25^x = 5$

$\Rightarrow x = \frac{1}{2}$

(b) $\log_5\left(\frac{1}{25}\right) = x$

$\Rightarrow 5^x = \frac{1}{25}$

$\Rightarrow x = -2$

There are two important results to remember:

$$\log_a 1 = 0$$
$$\log_a a = 1$$

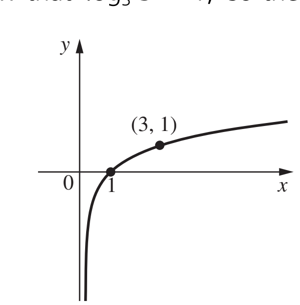
These come from $a^0 = 1$ and $a^1 = a$.

Example

Sketch the graph of $y = \log_3 x$.

We know the shape, and that the graph passes through (1, 0).

As the base is 3, we know that $\log_3 3 = 1$, so the graph passes through (3,1).



Exercise 2

1 Sketch these graphs.

a $y = \log_2 x$ **b** $y = \log_4 x$

c $y = \log_5 x$ **d** $y = \log_{10} x$

2 Sketch these functions on the same graph.

a $y = 3^x$ and $y = \log_3 x$ **b** $y = 5^x$ and $y = \log_5 x$

3 Without a calculator, evaluate these logarithms.

a $\log_2 4$ **b** $\log_3 9$ **c** $\log_5 25$ **d** $\log_3 27$ **e** $\log_6 216$

f $\log_2 32$ **g** $\log_4 64$ **h** $\log_2 64$ **i** $\log_8 64$ **j** $\log_{10} 100$

k $\log_{10} 1000$ **l** $\log_7 7$ **m** $\log_3 3$ **n** $\log_8 1$

4 Without a calculator, evaluate these logarithms.

a $\log_8 2$ **b** $\log_9 3$ **c** $\log_4 2$ **d** $\log_{25} 5$

e $\log_{64} 8$ **f** $\log_{64} 2$ **g** $\log_8 4$ **h** $\log_{16} 8$

5 Without a calculator, evaluate these logarithms.

a $\log_2\left(\frac{1}{16}\right)$ **b** $\log_2\left(\frac{1}{8}\right)$ **c** $\log_3\left(\frac{1}{9}\right)$ **d** $\log_3\left(\frac{1}{27}\right)$

e $\log_3\left(\frac{1}{81}\right)$ **f** $\log_8\left(\frac{1}{2}\right)$ **g** $\log_{25}\left(\frac{1}{5}\right)$

6 Without a calculator, evaluate these logarithms.

a $\log_a a$ **b** $\log_a a^2$ **c** $\log_a \sqrt{a}$ **d** $\log_a\left(\frac{1}{a}\right)$ **e** $-\log_a a$

5.3 Rules of logarithms

As there are for exponentials, there are rules for logarithms that help to simplify logarithmic expressions.

For exponentials, we have the rules:

1. $a^p \times a^q = a^{p+q}$ 2. $a^p \div a^q = a^{p-q}$ 3. $(a^p)^q = a^{pq}$

The corresponding rules for logarithms are:

1. $\log_a xy = \log_a x + \log_a y$ 2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a x^m = m \log_a x$

Proofs

1. Let $\log_a x = p$ and $\log_a y = q$.
   This means that $x = a^p$ and $y = a^q$.
   Since $xy = a^p \times a^q = a^{p+q}$, $xy = a^{p+q}$.
   By the definition of a logarithm this means that $\log_a xy = p + q$
   $$\Rightarrow \log_a xy = \log_a x + \log_a y$$

It is very important to remember that $\log_p a + \log_p b \neq \log_p(a + b)$.

2. Similarly, let $\log_a x = p$ and $\log_a y = q$.
   $$\text{So } \frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$$
   $$\text{Hence } \log_a\left(\frac{x}{y}\right) = p - q$$
   $$\Rightarrow \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

It is worth noting that $\frac{1}{\log_a x} = -\log_a x$.

3. Again, let $\log_a x = p$ and $\log_a y = q$.
   $$\text{So } x^m = (a^p)^m = a^{mp}$$
   $$\text{Hence } \log_a x^m = mp = m \log_a x$$

For these rules to work, the logarithms must have the same base.

Example

Simplify $\log_x 8 + \log_x 3 - \log_x 6$.

$$\log_x 8 + \log_x 3 - \log_x 6 = \log_x\left(\frac{8 \times 3}{6}\right) = \log_x 4$$

Example

Simplify $3 \log_p 2 - \log_p 12 + 2 \log_p 4$

$$\begin{aligned} 3 \log_p 2 - \log_p 12 + 2 \log_p 4 &= \log_p 2^3 - \log_p 12 + \log_p 4^2 \\ &= \log_p 8 - \log_p 12 + \log_p 16 \\ &= \log_p\left(\frac{8 \times 16}{12}\right) \\ &= \log_p\left(\frac{32}{3}\right) \end{aligned}$$

Example

Simplify and evaluate $2 \log_{10} 5 + 2 \log_{10} 2$.

$$\begin{aligned} 2 \log_{10} 5 + 2 \log_{10} 2 &= \log_{10} 5^2 + \log_{10} 2^2 \\ &= \log_{10} 25 + \log_{10} 4 \\ &= \log_{10} 100 \\ &= 2 \end{aligned}$$

These rules can also be used to solve equations involving logarithms.

Example

Solve $\log_6(x+2) + \log_6(x+1) = 1$ for $x > 0$.

$\log_6(x+2) + \log_6(x+1) = 1$

$\Rightarrow \log_6(x+2)(x+1) = 1$

$\Rightarrow \log_6(x+2)(x+1) = \log_6 6$ — Expressing 1 as a logarithm.

$\Rightarrow (x+2)(x+1) = 6$

$\Rightarrow x^2 + 3x + 2 = 6$

$\Rightarrow x^2 + 3x - 4 = 0$

$\Rightarrow (x+4)(x-1) = 0$

$\Rightarrow x = 1$ — Since $x > 0$.

Exercise 3

1 Simplify these.

a $\log_a 2 + \log_a 9$ **b** $\log_a 5 + \log_a 3$ **c** $\log_a 10 - \log_a 2$

d $\log_a 8 + \log_a 8$ **e** $\log_a 2 + \log_a 3 + \log_a 4$ **f** $\frac{1}{2}\log_a 16$

g $5\log_a 2$ **h** $2\log_a 6 + \log_a 2 - \log_a 12$ **i** $-3\log_a 2$

j $2\log_a 3 + 3\log_a 2$ **k** $\log_a 6 - 2\log_a 2 + \log_a 8$

2 Express each of these as a single logarithm of a number.

a $1 + \log_3 5$ **b** $\log_2 10 - 2$

c $5 - 2\log_2 6$ **d** $\log_a x + 2\log_a y - 3\log_a t$

3 Simplify these.

a $\log_{10} 4 + \log_{10} 125$ **b** $\log_3 63 - \log_3 7$

c $\log_6 2 + \log_6 3$ **d** $\log_4 36 - \log_4 18$

e $\log_3 6 + \log_3 12 - \log_3 8$ **f** $\log_6 12 - \log_6\left(\frac{1}{3}\right)$

g $\frac{1}{2}\log_2 16 - \frac{1}{3}\log_2 8$ **h** $\log_5 64 - 6\log_5 2$

i $\log_2 3 + \log_2 2 - \log_2 6 - \log_2 8$ **j** $\log_2\left(\frac{1}{4}\right) - 2\log_2\left(\frac{1}{8}\right)$

k $-2\log_4 8 + \log_4\left(\frac{1}{2}\right)$

4 Simplify these.

a $\log_a 3 + \log_a x + 2\log_a x$

b $\log_a 4 - \log_a 2x$

c $\log_a(x+1) - \log_a(x^2-1) + 2\log_a(x-1)$

d $3\log_a(x+2) - \log_a(3x^2-12) + \log_a(x-2)$

5 Simplify these.

a $\log_2 4$ **b** $\log_4 2$ **c** $\log_9 27$

d $\log_{27} 9$ **e** $\log_{10} 100$ **f** $\log_{100} 10$

What is the connection between $\log_x y$ and $\log_y x$?

6 If $\log_a y = \log_a 4 + 3\log_a x$, express y in terms of x.

7 If $\log_a y = 2\log_a 3 + 4\log_a x$, express y in terms of x.

8 If $\log_a y = \log_a p + 5\log_a x$, express y in terms of p and x.

9 If $2\log_2 y = \log_2(x+1) + 3$, show that $y^2 = 8(x+1)$.

10 Solve for $x > 0$.

a $\log_a x + \log_a 2 = \log_a 14$ **b** $\log_a x - \log_a 3 = \log_a 19$

c $\log_a x + 2\log_a 2 = \log_a 24$ **d** $\log_a x + 2\log_a 5 = \log_a 225$

e $\log_a x^2 + \log_a\left(\frac{1}{2}\right) = \log_a 32$ **f** $\log_a 6 + \log_a x = \log_a 1$

g $\frac{1}{2}\log_a x + \log_a 6 = \log_a 30$ **h** $2\log_a x - \log_a x = \log_a 9$

11 Solve for $x > 0$.

a $\log_a(x+2) + \log_a(x-1) = \log_a 4$ **b** $\log_5(x+1) + \log_5(x-3) = 1$

c $\log_4(3x-1) - \log_4(x-1) = 1$ **d** $\log_8(x^2-1) - \log_8(x-1) = 2$

e $\log_{16}(x+2) - \log_{16}(x-6) = \frac{1}{2}$ **f** $\log_7(2x+5) - \log_7(x-5) = \log_7\left(\frac{x}{2}\right)$

12 Volume of sound is measured in decibels. The difference in volume between two sounds can be calculated using the formula $d = 50\log_{10}\left(\frac{S_1}{S_2}\right)$ where S_1 and S_2 are sound intensities $(S_1 > S_2)$. The volume of normal conversation is 60 dB and the volume of a car horn is 110 dB. The sound intensity of normal conversation (S_2) is 40 phons. What is the sound intensity of a car horn (S_1)?

5.4 Logarithms on a calculator

The natural base

There is a special base, denoted e, which is known as the **natural base**. The reason why this base is special is covered in Chapter 9. This number e is the irrational number 2.718 …

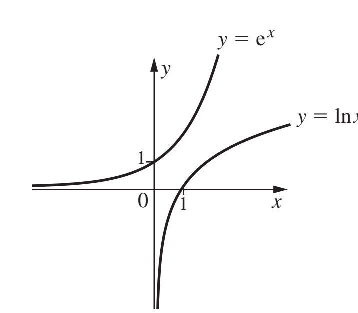
The exponential function to the base e is $f(x) = e^x$, which is also written $\exp(x)$.

$\exp(x) = e^x$

There is also a notation for its inverse (logarithmic) function known as the **natural logarithmic function**. It is also sometimes called a Naperian logarithm after John Napier.

$\log_e x = \ln x$

The graphs of these functions have the same shape as other logarithmic and exponential graphs.



In particular note that $\ln e = 1$.

Calculators perform logarithms in only two bases, 10 and e. For these two logarithms, the base is rarely explicitly stated. For the natural base the notation is ln, and for base 10 it is often just written log *x*, and the base is assumed to be 10.

Example

Find $\log_{10} 7$ and $\ln 7$.

```
log(7)
      .84509804
```

```
ln(7)
     1.945910149
```

Change of base formula

To find logarithms in other bases, we need to change the base using this formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof

$\log_a x = y$

$\Rightarrow x = a^y$

$\Rightarrow \log_b x = \log_b a^y$

$\Rightarrow \log_b x = y \log_b a$

$\Rightarrow y = \dfrac{\log_b x}{\log_b a}$

If we are using this formula to find a logarithm on a calculator, it is often written as

$$\log_a x = \frac{\ln x}{\ln a}$$

Example

Use the change of base formula to evaluate these.

(a) $\log_2 8$ (b) $\log_3 10$

```
ln(8)/ln(2)
              3
```

```
ln(10)/ln(3)
     2.095903274
```

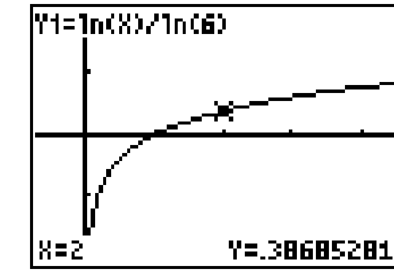
Be careful with brackets.

The change of base formula can be used to sketch any logarithmic function on the calculator.

Example

Sketch $y = \log_6 x$.

```
Plot1 Plot2 Plot3
\Y1=ln(X)/ln(6)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



Exercise 3 question 5 asked "What is the connection between $\log_x y$ and $\log_y x$?" The answer is a special case of the change of base formula, namely that

$$\log_x y = \frac{\log_y y}{\log_y x} = \frac{1}{\log_y x}$$

This can be used to help solve equations.

Example

Use the change of base formula to solve $\log_9 x + 2 \log_x 9 = 3$.

This can be changed into $\log_9 x + \dfrac{2}{\log_9 x} = 3$.

Multiplying by $\log_9 x$ gives

$$(\log_9 x)^2 + 2 = 3 \log_9 x$$
$$\Rightarrow (\log_9 x)^2 - 3 \log_9 x + 2 = 0$$
$$\Rightarrow y^2 - 3y + 2 = 0$$

Let $y = \log_9 x$

$$\Rightarrow (y - 1)(y - 2) = 0$$
$$\Rightarrow (\log_9 x - 1)(\log_9 x - 2) = 0$$
$$\Rightarrow \log_9 x = 1 \text{ or } \log_9 x = 2$$
$$\Rightarrow x = 9 \text{ or } x = 81$$

Exercise 4

1 Using a calculator, evaluate these.

a $\log_{10} 1000$ **b** $\log_{10} 8$ **c** $\log_{10} 26$

d $\log_{10} 3$ **e** $\log_{10}\left(\frac{1}{2}\right)$

2 Using a calculator, evaluate these.

a $\ln 10$ **b** $\ln 9$ **c** $\ln 31$

d $\ln\left(\frac{1}{2}\right)$ **e** $\ln 7.5$ **f** $\ln(0.328)$

3 Use the change of base formula to evaluate these.

a $\log_4 16$ **b** $\log_2 9$ **c** $\log_3 11$ **d** $\log_4 17$ **e** $\log_5 80$

f $\log_6 12$ **g** $\log_9 12$ **h** $\log_4 13$ **i** $\log_4\left(\frac{1}{2}\right)$ **j** $\log_5\left(\frac{1}{6}\right)$

k $\log_7\left(\frac{1}{8}\right)$ **l** $\log_8\left(\frac{1}{4}\right)$ **m** $\log_6\left(\frac{2}{3}\right)$ **n** $\log_5\left(\frac{5}{7}\right)$ **o** $\log_8 9.21$

p $\log_6 4.38$ **q** $\log_4(0.126)$ **r** $\log_9(0.324)$

4 Use your calculator to sketch these.

a $y = \log_5 x$ **b** $y = \log_7 x$ **c** $y = \log_9 x$

5 Use the change of base formula to solve these.

a $\log_4 x + 5\log_x 4 = 6$

b $\log_2 x - 6\log_x 2 = 1$

c $\log_7 x - 12\log_x 7 = 4$

5.5 Exponential equations

Logarithms can be used to solve exponential equations and this is one of their greatest applications today. When logarithms were first advanced by Napier, they were used as a computational aid. Exponential equations are ones where we are trying to find the power. The logarithmic rule of $\log_a x^p = p\log_a x$ is particularly useful for these equations.

Method for solving an exponential equation

1. Take natural logs of both sides.
2. "Bring down" the power.
3. Divide the logs.
4. Solve for x.

Example

Solve $2^x = 7$.

$$2^x = 7$$
$$\Rightarrow \ln 2^x = \ln 7$$
$$\Rightarrow x\ln 2 = \ln 7$$
$$\Rightarrow x = \frac{\ln 7}{\ln 2}$$
$$\Rightarrow x = 2.81$$

Example

Solve $e^x = 12$.

$$e^x = 12$$
$$\Rightarrow \ln e^x = \ln 12$$
$$\Rightarrow x\ln e = \ln 12$$
$$\Rightarrow x = \ln 12$$

Since $\ln e = 1$.

$$\Rightarrow x = 2.48$$

Example

Solve $3^{2x-1} = 40$.

$$3^{2x-1} = 40$$
$$\Rightarrow \ln 3^{2x-1} = \ln 40$$
$$\Rightarrow (2x-1)\ln 3 = \ln 40$$
$$\Rightarrow 2x - 1 = \frac{\ln 40}{\ln 3}$$
$$\Rightarrow 2x - 1 = 3.357\ldots$$
$$\Rightarrow x = 2.18 \text{ (3 sf)}$$

Natural logarithm equations can also be solved using a calculator.

Example

Solve $\ln x = 11$.

Remembering this means $\log_e x = 11$, this can be written

$$x = e^{11}$$
$$\Rightarrow x = 59\,900 \text{ (3 sf)}$$

Exponential functions are very important in the study of growth and decay, and are often used as mathematical models.

Example

A population of rats increases according to the formula $R(t) = 8e^{0.22t}$, where t is the time in months.
(a) How many rats were there at the beginning?
(b) How long will it be until there are 80 rats?

(a) When $t = 0$, $R(0) = 8e^0 = 8$

(b) $8e^{0.22t} = 80$

$$\Rightarrow e^{0.22t} = 10$$
$$\Rightarrow \ln e^{0.22t} = \ln 10$$
$$\Rightarrow 0.22t = \ln 10$$
$$\Rightarrow t = \frac{\ln 10}{0.22}$$
$$\Rightarrow t = 10.5 \text{ months}$$

Example

For a radioactive isotope $A = A_0e^{-kt}$, where A is the mass of isotope in grams, A_0 is the initial mass, and t is time in years.
In 5 years, 40 g of this substance reduced to 34 g.
(a) Find the value of k, correct to 3 sig figs.
(b) Find the half-life of this substance.

(a) $A_0 = 40, A = 34, t = 5$

$$\Rightarrow 40e^{-5k} = 34$$
$$\Rightarrow e^{-5k} = \frac{34}{40}$$
$$\Rightarrow -5k = \ln\left(\frac{34}{40}\right)$$
$$\Rightarrow k = 0.0325$$

(b) The half-life of a radioactive substance is the time taken for only half of the original amount to remain.

i.e. $\frac{A}{A_0} = \frac{1}{2}$

$$\Rightarrow e^{-0.0325t} = \frac{1}{2}$$
$$\Rightarrow -0.0325t = \ln\frac{1}{2}$$
$$\Rightarrow t = 21.3 \text{ years}$$

Example

Solve $(6^x)(3^{2x+1}) = 4^{x+2}$, giving your answer in the form $x = \frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Q}$.

$$(6^x)(3^{2x+1}) = 4^{x+2}$$
$$\Rightarrow \ln[(6^x)(3^{2x+1})] = \ln 4^{x+2}$$
$$\Rightarrow x\ln 6 + (2x+1)\ln 3 = (x+2)\ln 4$$
$$\Rightarrow x\ln 6 + 2x\ln 3 + \ln 3 = x\ln 4 + 2\ln 4$$
$$\Rightarrow x\ln 6 + x\ln 9 + \ln 3 = x\ln 4 + \ln 16$$
$$\Rightarrow x(\ln 6 + \ln 9 - \ln 4) = \ln 16 - \ln 3$$
$$\Rightarrow x\ln\left(\frac{27}{2}\right) = \ln\left(\frac{16}{3}\right)$$
$$\Rightarrow x = \frac{\ln\left(\frac{16}{3}\right)}{\ln\left(\frac{27}{2}\right)}$$

Sometimes exponential equations can be reduced to quadratic form.

Example

Solve $7(3^{x+1}) = 2 + \frac{3}{3^x}$, giving the answer in the form $a - \log_3 b$, where $a, b \in \mathbb{Z}$.

$$7(3^{x+1}) = 2 + \frac{3}{3^x}$$
$$\Rightarrow 7(3^x)(3^1) = 2 + \frac{3}{3^x}$$

$$\Rightarrow 21(3^x) = 2 + \frac{3}{3^x}$$
$$\Rightarrow 21(3^{2x}) = 2(3^x) + 3$$
$$\Rightarrow 21(3^{2x}) - 2(3^x) - 3 = 0$$
$$\Rightarrow 21y^2 - 2y - 3 = 0$$

Let $y = 3^x$.

$$\Rightarrow (7y-3)(3y+1) = 0$$
$$\Rightarrow y = \frac{3}{7} \text{ or } y = -\frac{1}{3}$$
$$\Rightarrow 3^x = \frac{3}{7}$$
$$\Rightarrow x = \log_3\left(\frac{3}{7}\right)$$
$$\Rightarrow x = 1 - \log_3 7$$

Exercise 5

1 Solve for x.

a $2^x = 256$ **b** $3^x = 40$ **c** $5^x = 20$

d $12^x = 6500$ **e** $8^x = 6$ **f** $14^x = 3$

2 Solve for x.

a $e^x = 12$ **b** $e^x = 30$ **c** $e^x = 270$

d $4e^x = 18$ **e** $8e^x = 3$

3 Solve for x.

a $\ln x = 9$ **b** $\ln x = 2$ **c** $\ln x = 10$

d $\ln x = 16$ **e** $\ln x = 0.2$

4 Find the least positive value of $x \in \mathbb{Z}$ for which the inequality is true.

a $2^x > 350$ **b** $3^x > 300$ **c** $10^x > 2^{10}$ **d** $5^x > 7200$

5 The number of bacteria in a culture is given by $B(t) = 40e^{0.6t}$, where t is the time in days.

a How many bacteria are there when $t = 0$?

b How many bacteria are there after 2 days?

c How long will it take for the number of bacteria to increase to ten times its original number?

6 According to one mobile phone company, the number of people owning a mobile phone is growing according to the formula $N(t) = 100\,000e^{0.09t}$, where t is time in months. Their target is for 3 million people to own a mobile phone. How long will it be before this target is reached?

7 When a bowl of soup is removed from the microwave, it cools according to the model $T(t) = 80e^{-0.12t}$, t in minutes and T in °C.

a What was its temperature when removed from the microwave?

b The temperature of the room is 22°C. How long will it be before the soup has cooled to room temperature?

8 A radioactive isotope is giving off radiation and hence losing mass according to the model $M(t) = 2100e^{-0.012t}$, t in years and M in grams.

a What was its original mass?

b What will its mass be after 20 years?

c What is the half-life of the isotope?

9 The height of a satellite orbiting Earth is changing according to the formula $H(t) = 30000e^{-0.2t}$, t in years and H in km.

a What will be the height of the satellite above the Earth after 2 years?

b When the satellite reaches 320 km from the Earth, it will burn up in the Earth's atmosphere. How long before this happens?

10 Scientists are concerned about the population of cheetahs in a game park in Tanzania. Their study in 2006 produced the model $P(t) = 220e^{-0.15t}$, where P is the population of cheetahs and t is the time in years from 2006.

a How many cheetahs were there in 2006?

b How many cheetahs do they predict will be in the park in 2015?

c If the population drops to single figures, scientists predict the remaining cheetahs will not survive. When will this take place?

11 The pressure in a boiler is falling according to the formula $P_t = P_0e^{-kt}$, where P_0 is the initial pressure, P_t is the pressure at time t, and t is the time in hours.

a At time zero, the pressure is 2.2 units but 24 hours later it has dropped 1.6 units. Find the value of k to 3 sf.

b If the pressure falls below 0.9 units, the boiler cuts out. How long before it will cut out?

c If the boiler's initial pressure is changed to 2.5 units, how much longer will it be operational?

12 A radioactive substance is losing mass according to the formula $M_t = M_0e^{-kt}$ where M_0 is the initial mass, M_t is the mass after t years.

a If the initial mass is 900 g and after 5 years it has reduced to 850 g, find k.

b What is the half-life of this substance?

13 Solve $3^{x+1} = 2^{2-x}$. Give your answer in the form $x = \frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Z}$.

14 Solve $(4^x)(3^{2x+1}) = 6^{x+1}$. Give your answer in the form $x = \frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Z}$.

15 Solve $(4^x)(5^{x+1}) = 2^{2x+1}$. Give your answer in the form $x = \frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Q}$.

16 Solve $(2^x)(3^{2x+1}) = 4^{x+3}$. Give your answer in the form $x = \frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Q}$.

17 Solve $5(2^{x+1}) = 3 + \frac{4}{2^x}$, giving your answer in the form $x = a - \log_2 b$, where $a, b \in \mathbb{Z}$.

18 Solve $2(4^{x+1}) = 2 + \frac{3}{4^x}$, giving your answer in the form $x = a - \log_4 b$, where $a, b \in \mathbb{Z}$.

19 Solve $2(4^{x+2}) = 12 + \frac{5}{4^x}$, giving your answer in the form $x = a - \log_4 b$, where $a, b \in \mathbb{Q}$.

20 Solve $3(6^{x-1}) = 1 + \frac{4}{6^x}$, giving your answer in the form $x = a - \log_6 b$, where $a, b \in \mathbb{Z}$.

21 Solve $4^x + 4(2^x) - 5 = 0$.

22 Solve $9^x + 4(3^x) - 12 = 0$.

5 Exponential and Logarithmic Functions

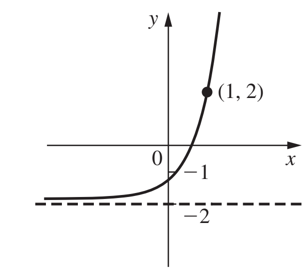
5.6 Related graphs

Exponential and logarithmic graphs are transformed in the same way as other functions, as studied previously.

We can sketch and interpret related exponential and logarithmic graphs using this information.

Example

Sketch the graph of $y = 4^x - 2$.

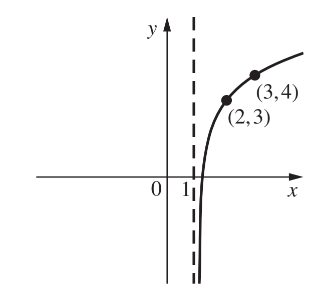


This is a vertical shift downwards of 2 units.

For exponential graphs, we often plot the points for $x = 0$ and $x = 1$.

Example

Sketch the graph of $y = \log_2(x - 1) + 3$.



This is a horizontal shift right of 1 unit and a vertical shift of 3 units.

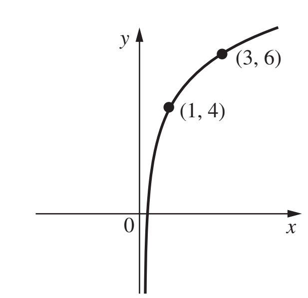
For logarithmic graphs, we often plot the points when $x = 1$ and $x = a$, where a is the base (or their images under transformation). So here $(1, 0) \rightarrow (2, 3)$ and $(2, 1) \rightarrow (3, 4)$

Notice the vertical asymptote has moved to $x = 1$.

Example

Sketch the graph of $y = \log_3 x^2 + 4$.

From log rules, we know that this is the same as $y = 2\log_3 x + 4$.

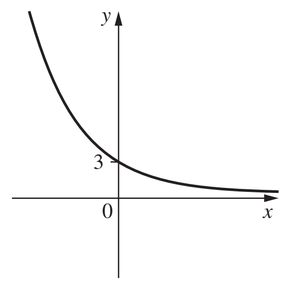


$(1, 0) \rightarrow (1, 4)$

$(3, 1) \rightarrow (3, 6)$

Example

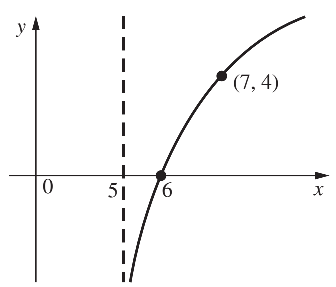
For this graph of $y = ke^{-x}$, what is the value of k?



As (0,3) lies on the graph it has been stretched $\times 3$. So $k = 3$.

Example

Part of the graph of $y = p\log_2(x - q)$ is shown. What are the values of p and q?



The graph has been shifted 5 places right so $q = 5$.

So for (7, 4), $4 = p\log_2(7 - 5)$
$= p\log_2 2$
$= p$

So $p = 4$ and $q = 5$

Exercise 6

1 Sketch these graphs.
a $y = 2^x$ **b** $y = 2^x - 3$ **c** $y = 2^{-x}$ **d** $y = 2^{x+3}$

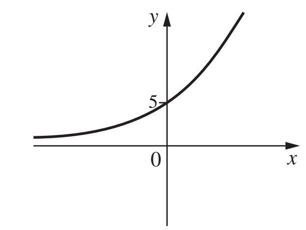
2 Sketch these graphs.
a $y = e^x$ **b** $y = 4e^x$ **c** $y = -e^x$ **d** $y = e^{x-2}$

3 Sketch these graphs.
a $y = 4e^{-x}$ **b** $y = 2e^x - 1$ **c** $y = 3e^{x+2}$ **d** $y = 5e^{x-1} + 3$

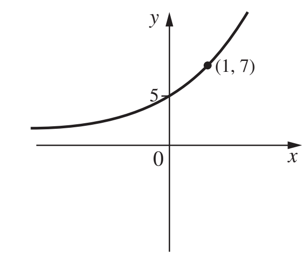
4 Sketch these graphs.
a $y = \log_4 x$ **b** $y = \log_4 x + 2$
c $y = \log_4(x + 2)$ **d** $y = -\log_4 x$

5 Sketch these graphs.
a $y = \log_3 x$ **b** $y = \log_3 x^2$
c $y = \log_3\left(\frac{1}{x}\right)$ **d** $y = 3\log_3(x + 2)$

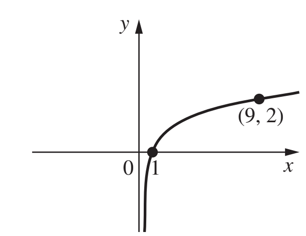
6 For this graph of $y = ke^x$, what is the value of k?



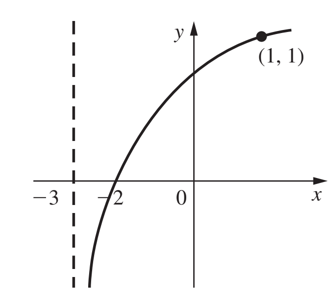
7 For this graph of $y = k \times 2^x + p$, what are the values of k and p?



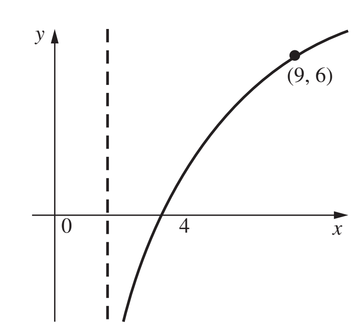
8 The sketch shows the graph of $y = \log_a x$. Find the value of a.



9 The sketch shows the graph of $y = \log_a(x + p)$. Find the values of p and a.



10 The sketch shows part of the graph of $y = a\log_6(x - p)$. Find the values of a and p.



Review exercise

1 Simplify these.

a $\dfrac{x^7 \times x^2}{x^3}$ **b** $\dfrac{5p^{\frac{1}{2}} \times 3p^{\frac{3}{4}}}{p^{\frac{3}{2}}}$ **c** $x^{-\frac{1}{2}}(2x^{\frac{1}{2}} + 4x^{-\frac{3}{2}})$

2 Draw these graphs.

a $y = 6^x$ **b** $\log_6 x$

3 Evaluate these.

a $\log_2 32$ **b** $\log_5 125$ **c** $\log_8 2$ **d** $\log_5\left(\frac{1}{25}\right)$

4 Simplify these.

a $\log_a 16 + \log_a 3$ **b** $\log_p 8 + \log_p 4 - \log_p 16$ **c** $2 \log_a 5 + 1$

5 Simplify these.

a $\log_3 x + \log_3 8$ **b** $\log_4 6 + \log_4 2 - \log_4 3$

6 Solve for $x > 0$.

a $\log_a x + \log_a 6 = \log_a 54$ **b** $\frac{1}{2}\log_a x + \log_a 5 = \log_a 45$

c $\log_5(x - 1) + \log_5(x + 2) = \log_5 10$ **d** $\log_3(x + 2) - \log_3(x - 1) = 2$

7 Evaluate these.

a $\log_4 7$ **b** $\log_9 4$ **c** $\log_{11} 2$ **d** $3 \log_8 5$

8 Solve for x.

a $\ln x = 9$ **b** $\ln x = 17$ **c** $5 \ln x = 19$

9 Solve for x.

a $3^x = 320$ **b** $7^x = 2$ **c** $e^x = 8$ **d** $5e^x = 19$

10 Find the least positive value of $x \in \mathbb{Z}$ for which the inequality is true.

a $3^x > 190$ **b** $5^x > 2000$ **c** $e^x > 291$

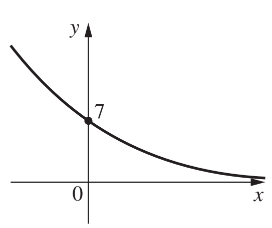
11 Sketch these graphs.

a $y = 5^x - 2$ **b** $y = 6^{x+3}$ **c** $y = 4^{x-1} + 2$ **d** $y = -3^x - 5$

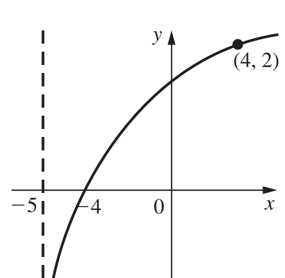
12 Sketch these graphs.

a $y = \log_5 x - 2$ **b** $y = \log_7\left(\frac{1}{x}\right)$ **c** $y = 5 \log_2(x + 1)$

13 For this graph of $y = ke^{-x}$, what is the value of k?



14 The sketch shows part of the graph of $y = \log_p(x + q)$. What are the values of p and q?



15 A truck has a slow puncture in one of its tyres, causing the pressure to drop. The pressure at time t, P_t, is modelled by $P_t = P_0e^{-kt}$, where t is in hours and P_0 is the inflation pressure.

a Initially the tyre is inflated to 50 units. After 18 hours, it drops to 16 units. Calculate the value of k.

b The truck will not be allowed to make a journey if the pressure falls below 30 units. If the driver inflates the tyre to 50 units immediately before departure, will he be able to make a round trip that takes 6 hours?

16 Solve $\log_{16} \sqrt[3]{100 - x^2} = \frac{1}{2}$. [IB Nov 03 P1 Q10]

17 Find the exact value of x satisfying the equation $(3^x)(4^{2x+1}) = 6^{x+2}$

Give your answer in the form $\frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Z}$. [IB May 03 P1 Q12]

18 Solve $2(5^{x+1}) = 1 + \frac{3}{5^x}$, giving your answer in the form $a + \log_5 b$,

where $a, b \in \mathbb{Z}$. [IB Nov 03 P1 Q19]

19 Solve the simultaneous equations $\log_x y = 1$ and $xy = 16$ for $x, y > 0$.

20 Solve the simultaneous equations $\log_a(x + y) = 0$ and $2 \log_a x = \log(4y + 1)$.

21 Solve the system of simultaneous equations:

$$x + 2y = 5$$
$$4^x = 8^y$$

[IB Nov 98 P1 Q2]

22 If $f(x) = \ln(6x^2 - 5x - 6)$, find

a the exact domain of $f(x)$

b the range of $f(x)$. [IB Nov 98 P1 Q7]

23 Find all real values of x so that $3^{x^2-1} = (\sqrt{3})^{126}$. [IB May 98 P1 Q3]

24 a Given that $\log_a b = \frac{\log_c b}{\log_c a}$, find the real numbers k and m such that

$\log_9 x^3 = k \log_3 x$ and $\log_{27} 512 = m \log_3 8$.

b Find all values of x for which $\log_9 x^3 + \log_3 x^{\frac{1}{2}} = \log_{27} 512$.

[IB Nov 97 P1 Q4]