15 Integration 2 – Further Techniques

Johann Bernoulli, Jakob's brother, was born on 27 July 1667 and was the tenth child of Nicolaus and Margaretha Bernoulli. Johann's father wished him to enter the family business, but this did not suit Johann and in the end he entered the University of Basel to study medicine. However, he spent a lot of time studying mathematics with his brother, Jakob, as his teacher. He worked on Leibniz's papers on calculus and within two years he had become the equal of his brother in mathematical skill, and he moved to Paris where he worked with de l'Hôpital. He then returned to Basel and at this stage Johann and Jakob worked together and learned much from each other. However this was not to last and their friendly rivalry descended into open hostility over the coming years. Among Johann's many mathematical achievements were work on the function $y = x^x$, and investigating series using the method of integration by parts. In this chapter we will use techniques that treat integration as the reverse of differentiation and this is exactly how Johann worked with it. His great success in mathematics was rewarded when in 1695 the he accepted the offer of the chair of mathematics at the University of Groningen. This gave him equal status to his brother Jakob who was becoming increasingly jealous of Johann's progress. During the ten years he spent at Groningen the battle between the two brothers escalated. In 1705 he left Groningen to return to Basel unaware that his brother had died two days previously. Ironically, soon after his return to Basel he was offered his brother's position at the University of Basel, which he accepted. He stayed there until his death on 1 January 1748.

Johann Bernoulli

Function $f(x)$	Integral + c $\int f(x)\,dx$
x^n	$\frac{1}{n+1}x^{n+1}$
e^x	e^x
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\sec x \tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{-1}{\sqrt{1-x^2}}$	$\cos^{-1} x$
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1}\frac{x}{a}$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
a^x	$\frac{a^x}{\ln\lvert a\rvert}$

Example

$$\int \sin 2x\,dx = -\frac{\cos 2x}{2} + c$$

Example

$$\int \frac{1}{2x-1}\,dx = \frac{1}{2}\ln\lvert 2x-1\rvert + c$$

In this chapter we will look at the techniques of integrating more complicated functions and a wider range of functions. Below is the complete list of basic results.

These are as a result of inspection and were dealt with in Chapter 14.

15.1 Integration as a process of anti-differentiation – direct reverse

For more complex questions, the inspection method can still be used. We call this method **direct reverse**.

Method of direct reverse

1. Decide what was differentiated to get the function in the question and write it down ignoring the constants.
2. Differentiate this.
3. Divide or multiply by constants to find the required form.

Example

$$\int e^{2x}\,dx$$

1. We begin with $y = e^{2x}$.
2. $\frac{dy}{dx} = 2e^{2x}$
3. So $2\int e^{2x}\,dx = e^{2x}$, therefore $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$

Example

$$\int 4^{2x}\,dx$$

1. We begin with $y = 4^{2x}$.
2. $\frac{dy}{dx} = \ln 4 \times 4^{2x} \times 2 = 2\ln 4 \cdot 4^{2x}$
3. So $2\ln 4\int 4^{2x}\,dx = 4^{2x}$, therefore $\int 4^{2x}\,dx = \frac{4^{2x}}{2\ln 4} + c$

Example

$$\int \cos\left(4\theta - \frac{\pi}{3}\right)d\theta$$

1. We begin with $y = \sin\left(4\theta - \frac{\pi}{3}\right)$.
2. $\frac{dy}{d\theta} = 4\cos\left(4\theta - \frac{\pi}{3}\right)$
3. $4\int \cos\left(4\theta - \frac{\pi}{3}\right)d\theta = \sin\left(4\theta - \frac{\pi}{3}\right)$,

   therefore $\int \cos\left(4\theta - \frac{\pi}{3}\right)d\theta = \frac{1}{4}\sin\left(4\theta - \frac{\pi}{3}\right) + c$

The method of direct reverse can also prove all the basic results given at the start of the chapter.

Example

$\int a^x \, dx$

1. We begin with $y = a^x$.
2. $\frac{dy}{dx} = a^x \ln|a|$
3. $\ln|a| \int a^x \, dx = a^x$, therefore $\int a^x \, dx = \frac{a^x}{\ln|a|} + c$

This technique allows us to do much more complicated examples including some products and quotients.

Integration of products and quotients using direct reverse

Unlike differentiation this is not quite so simple. Considering differentiation, an answer to a derivative may be a product or a quotient from more than one technique, e.g. chain rule, product rule or quotient rule. This means there has to be more than one way to integrate products and quotients. Here cases which can be done by direct reverse will be considered.

Products

This occurs when one part of the product is a constant multiplied by the derivative of the inside function. If we are integrating $f(x) \times gh(x)$, then $f(x) = ah'(x)$ where a is a constant for the technique to work.

This may seem complicated but it is easy to apply and is a quick way of doing some quite advanced integration.

Example

$\int x^2(x^3 + 3)^3 \, dx$

Direct reverse works in this case, since if $h(x) = x^3 + 3$, $h'(x) = 3x^2$, then $a = \frac{1}{3}$.

The value of a is not used in this method, but it is necessary to check that a constant exists for the method to work.

1. This begins with $y = (x^3 + 3)^4$.
2. $\frac{dy}{dx} = 4(x^3 + 3)^3 \cdot 3x^2 = 12x^2(x^3 + 3)^3$
3. $12\int x^2(x^3 + 3)^3 \, dx = (x^3 + 3)^4$,

   therefore $\int x^2(x^3 + 3)^3 \, dx = \frac{1}{12}(x^3 + 3)^4 + c$

Now consider the example $\int x^2(2x - 1)^4 \, dx$. In this case $h(x) = 2x - 1$ and hence $h'(x) = 2$. Since $f(x) = x^2$, then $f(x) \neq ah'(x)$ and hence this technique cannot be

used. Further techniques for integrating products and quotients which will deal with this will be discussed later in the chapter.

This technique can never be made to work by letting a be a function of x.

Example

$\int \sin x\sqrt{1 + \cos x} \, dx$

The method works in this case, since if $h(x) = 1 + \cos x$, $h'(x) = -\sin x$ then $a = -1$. We begin by writing the integral in the form $\int \sin x(1 + \cos x)^{\frac{1}{2}} \, dx$.

1. This begins with $y = (1 + \cos x)^{\frac{3}{2}}$.
2. $\frac{dy}{dx} = \frac{3}{2}(1 + \cos x)^{\frac{1}{2}}(-\sin x)$
3. $\frac{-3}{2}\int \sin x(1 + \cos x)^{\frac{1}{2}} \, dx = (1 + \cos x)^{\frac{3}{2}}$

   therefore $\int \sin x(1 + \cos x)^{\frac{1}{2}} \, dx = -\frac{2}{3}(1 + \cos x)^{\frac{3}{2}} + c$

Sometimes the examples can be somewhat disguised.

Example

$\int (x + 1)(x^2 + 2x - 5)^6 \, dx$

It still works in this case, since if $h(x) = x^2 + 2x - 5$, $h'(x) = 2x + 2 = 2(x + 1)$, then $a = \frac{1}{2}$.

1. This begins with $y = (x^2 + 2x - 5)^7$.
2. $\frac{dy}{dx} = 7(x^2 + 2x - 5)^6(2x + 2) = 14(x + 1)(x^2 + 2x - 5)^6$
3. $14\int (x + 1)(x^2 + 2x - 5)^6 \, dx = (x^2 + 2x - 5)^7$,

   therefore $\int (x + 1)(x^2 + 2x - 5)^6 \, dx = \frac{1}{14}(x^2 + 2x - 5)^7 + c$

Example

$\int (2x + 1)e^{2x^2 + 2x} \, dx$

The method works here since if $h(x) = 2x^2 + 2x$, $h'(x) = 4x + 2 = 2(2x + 1)$, then $a = \frac{1}{2}$.

1. This begins with $y = e^{2x^2 + 2x}$.
2. $\frac{dy}{dx} = (4x + 2)e^{2x^2 + 2x} = 2(2x + 1)e^{2x^2 + 2x}$
3. $2\int (2x + 1)e^{2x^2 + 2x} \, dx = e^{2x^2 + 2x}$,

   therefore $\int (2x + 1)e^{2x^2 + 2x} \, dx = \frac{1}{2}e^{2x^2 + 2x} + c$

Quotients

This occurs when the numerator of the quotient is in the form of a constant multiplied by the derivative of the inside function.

If we are integrating $\frac{f(x)}{gh(x)}$ then $f(x) = ah'(x)$ for the technique to work. The quotients often come in the form $\frac{f'(x)}{f(x)}$ which was met in Chapter 9 when differentiating logarithmic functions. This came from the form $y = \ln(f(x))$.

Example

$$\int \frac{x^2}{1+x^3}dx$$

Direct reverse works in this case, since if $h(x) = 1 + x^3$, $h'(x) = 3x^2$ then $a = \frac{1}{3}$.

1. This begins with $y = \ln|1 + x^3|$.
2. $\frac{dy}{dx} = \frac{3x^2}{1+x^3}$
3. $3\int \frac{x^2}{1+x^3}dx = \ln|1 + x^3|$, therefore $\int \frac{x^2}{1+x^3}dx = \frac{1}{3}\ln|1 + x^3| + c$

Example

$$\int \frac{\sin x}{22 - \cos x}dx$$

The method works, since if $h(x) = 22 - \cos x$, $h'(x) = \sin x$ then $a = 1$.

1. This begins with $y = \ln|22 - \cos x|$.
2. $\frac{dy}{dx} = \frac{\sin x}{22 - \cos x}$
3. Therefore $\int \frac{\sin x}{22 - \cos x}dx = \ln|22 - \cos x| + k$

Example

$$\int \frac{\sin x}{\cos x}dx$$

The method works, since if $h(x) = \cos x$, $h'(x) = -\sin x$ then $a = -1$.

1. This begins with $y = \ln|\cos x|$.
2. $\frac{dy}{dx} = \frac{-\sin x}{\cos x}$
3. $-\int \frac{\sin x}{\cos x}dx = \ln|\cos x|$ therefore $\int \frac{\sin x}{\cos x}dx = -\ln|\cos x| + c$

This is the method of integrating tan x.

Example

$$\int \frac{3e^x}{14 + e^x}dx$$

Again this works, since if $h(x) = 14 + e^x$, $h'(x) = e^x$ then $a = 1$.

1. This begins with $y = \ln|14 + e^x|$.
2. $\frac{dy}{dx} = \frac{e^x}{14 + e^x}$
3. Therefore $\int \frac{e^x}{14 + e^x}dx = \ln|14 + e^x|$

$$\Rightarrow \int \frac{3e^x}{14 + e^x}dx = 3\int \frac{e^x}{14 + e^x}dx = 3\ln|14 + e^x| + c$$

This is a slightly different case since the constant is part of the question rather than being produced through the process of integration. However, it is dealt with in the same way.

There is a danger of assuming that all integrals of quotients become natural logarithms. Many are, but not all, so care needs to be taken. The following examples demonstrate this.

Example

$$\int_2^p \frac{x^2}{(x^3+3)^4}dx$$

The technique will still work in the case of definite integration, since if $h(x) = x^3 + 3$, $h'(x) = 3x^2$ then $a = \frac{1}{3}$. We begin by writing the integral in the form $\int_2^p x^2(x^3 + 3)^{-4}dx$.

1. This begins with $y = (x^3 + 3)^{-3}$.
2. $\frac{dy}{dx} = -3(x^3 + 3)^{-4} \times 3x^2 = -9x^2(x^3 + 3)^{-4}$
3. $-9\int_2^p x^2(x^3+3)^{-4}dx = [(x^3+3)^{-3}]_2^p$ therefore $\int_2^p x^2(x^3+3)^{-4}dx$

$$= \left[-\frac{1}{9}(x^3 + 3)^{-3}\right]_2^p$$

$$= \left[-\frac{1}{9}(p^3 + 3)^{-3} + \frac{1}{9}(8 + 3)^{-3}\right]$$

$$= \left[-\frac{1}{9(p^3 + 3)^3} + \frac{1}{11\,979}\right]$$

If the limits were both numbers, and an exact answer was not required, then a calculator could be used to evaluate the integral. However, if one or both of the limits are not known, an exact answer is required, or if the question appears on the non-calculator paper, then this technique must be used.

If the question had asked for an exact answer to $\int_2^3 \frac{x^2}{(x^3+3)^4}dx$, then we would proceed as above, but the final lines would be:

$$= \left[-\frac{1}{9}(27+3)^{-3} + \frac{1}{9}(8+3)^{-3}\right]$$
$$= \left[-\frac{1}{243\,000} + \frac{1}{11\,979}\right]$$
$$= \frac{254\,979}{2\,910\,897\,000}$$

Example

$$\int \frac{x}{\sqrt{x^2-1}}\,dx$$

The method works in this case, since if $h(x) = x^2 - 1$, $h'(x) = 2x$ then $a = \frac{1}{2}$.

$$\int x(x^2-1)^{-\frac{1}{2}}\,dx$$

1. This begins with $y = (x^2-1)^{\frac{1}{2}}$.
2. $\frac{dy}{dx} = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x) = x(x^2-1)^{-\frac{1}{2}}$
3. Therefore $\int x(x^2-1)^{-\frac{1}{2}}\,dx = (x^2-1)^{\frac{1}{2}} + c$.

Exercise 1

Without using a calculator, find the following integrals.

1 $\int \sin\left(\theta - \frac{3\pi}{4}\right) d\theta$ **2** $\int \cos\left(3x + \frac{\pi}{4}\right) dx$

3 $\int e^{32x-7}\,dx$ **4** $\int_0^2 2\cos x e^{\sin x}\,dx$

5 $\int \frac{2}{8x-9}\,dx$ **6** $\int x^5(x^6-9)^8\,dx$

7 $\int_0^p x^3(2x^4+1)\,dx$ **8** $\int x\sqrt{1+x^2}\,dx$

9 $\int x\sqrt{4x^2-3}\,dx$ **10** $\int \sec^2 x(3\tan x + 4)^3\,dx$

11 $\int \frac{x}{\sqrt{x^2-1}}\,dx$ **12** $\int_0^{0.5} \sin x(2\cos x - 1)^4\,dx$

13 $\int_0^a e^{2x}(6e^{2x}-7)\,dx$ **14** $\int \sin 2x\sqrt{\cos 2x - 1}\,dx$

15 $\int (x+1)(x^2+2x-4)^5\,dx$ **16** $\int \frac{2x-3}{x^2-3x+5}\,dx$

17 $\int \frac{3\sin x}{(\cos x + 8)^3}\,dx$ **18** $\int \frac{e^x}{2e^x - 4}\,dx$

19 $\int \frac{\cos x}{3\sin x - 12}\,dx$ **20** $\int (3x^2+2)(3x^3+6x-19)^{\frac{3}{2}}\,dx$

21 $\int \sin 2x(1-3\cos 2x)^{\frac{5}{2}}\,dx$ **22** $\int \frac{\sec^2 2x}{3\tan 2x - 7}\,dx$

23 $\int_0^1 \frac{x}{(x^2+1)^5}\,dx$ **24** $\int \frac{2x-1}{3x^2-3x+4}\,dx$

25 $\int \frac{2x-1}{(3x^2-3x+4)^4}\,dx$ **26** $\int_1^p \frac{2\ln x}{x}\,dx$

27 $\int \frac{2e^x}{e^x + e^{-x}}\,dx$ **28** $\int_1^p \frac{2x+1}{6x^2+6x-15}\,dx$

15.2 Integration of functions to give inverse trigonometric functions

Questions on inverse trigonometric functions come in a variety of forms. Sometimes the given results can be used as they stand and in other cases some manipulation needs to be done first. In more difficult cases the method of direct reverse needs to be used.

Example

$$\int \frac{1}{\sqrt{4-x^2}}\,dx$$

Using the result $\frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

$$\Rightarrow \int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + c$$

Example

$$\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx.$$

This situation it is not given in the form $\frac{1}{\sqrt{a^2-x^2}}$ and hence there are two options. It can either be rearranged into that form or alternatively we can use the method of direct reverse.

Rearranging gives:

$$\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx = \int \frac{1}{\sqrt{\frac{1}{4}}\sqrt{4-x^2}}\,dx$$
$$= \int \frac{2}{\sqrt{4-x^2}}\,dx$$
$$= 2\sin^{-1}\frac{x}{2} + c$$

Using direct reverse we begin with $y = \sin^{-1}\frac{x}{2}$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \frac{x^2}{4}}} \times \frac{1}{2}$$

Therefore $\displaystyle\int \frac{1}{2\sqrt{1 - \frac{x^2}{4}}}\,dx = \sin^{-1}\frac{x}{2}$

$$\Rightarrow \int \frac{1}{\sqrt{1 - \frac{x^2}{4}}}\,dx = 2\sin^{-1}\frac{x}{2} + c$$

The same thing happens with the inverse tan function.

Example

$$\int \frac{1}{1 + \frac{x^2}{9}}\,dx$$

Rearranging to the standard result of $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ gives:

$$\int \frac{1}{1 + \frac{x^2}{9}}\,dx = \int \frac{1}{\frac{1}{9}(9 + x^2)}\,dx$$

$$= \int \frac{9}{9 + x^2}\,dx$$

$$= \frac{9}{3}\tan^{-1}\frac{x}{3} + c$$

$$= 3\tan^{-1}\frac{x}{3} + c$$

Using direct reverse we begin with $y = \tan^{-1}\frac{x}{3}$.

$$\frac{dy}{dx} = \frac{1}{1 + \frac{x^2}{9}} \times \frac{1}{3}$$

Therefore $\displaystyle\int \frac{1}{3\left(1 + \frac{x^2}{9}\right)}\,dx = \tan^{-1}\frac{x}{3}$

$$\Rightarrow \int \frac{1}{\left(1 + \frac{x^2}{9}\right)}\,dx = 3\tan^{-1}\frac{x}{3} + c$$

Now we need to look at more complicated examples. If it is a $\frac{number}{\sqrt{quadratic}}$ or a $\frac{number}{quadratic}$, then we need to complete the square and then use the method of direct reverse. It should be noted that this is not the case for every single example as they could integrate in different ways, but this is beyond the scope of this syllabus.

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Example

$$\int \frac{1}{x^2 + 2x + 5}\,dx$$

Completing the square gives:

$$\int \frac{1}{x^2 + 2x + 5}\,dx = \int \frac{1}{(x + 1)^2 + 4}\,dx$$

$$= \int \frac{1}{4 + (x + 1)^2}\,dx$$

$$= \frac{1}{4}\int \frac{1}{1 + \left(\frac{x + 1}{2}\right)^2}\,dx$$

Using direct reverse, we begin with $y = \tan^{-1}\left(\frac{x + 1}{2}\right)$.

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x + 1}{2}\right)^2} \times \frac{1}{2}$$

Therefore $\displaystyle\frac{1}{2}\int \frac{1}{1 + \left(\frac{x + 1}{2}\right)^2}\,dx = \tan^{-1}\left(\frac{x + 1}{2}\right)$

$$\Rightarrow \frac{1}{4}\int \frac{1}{1 + \left(\frac{x + 1}{2}\right)^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x + 1}{2}\right) + c$$

Example

$$\int \frac{1}{\sqrt{-x^2 - 4x + 12}}\,dx$$

Completing the square gives:

$$\int \frac{1}{\sqrt{-x^2 - 4x + 12}}\,dx = \int \frac{1}{\sqrt{-(x^2 + 4x - 12)}}\,dx$$

$$= \int \frac{1}{\sqrt{-[(x + 2)^2 - 16]}}\,dx$$

$$= \int \frac{1}{\sqrt{16 - (x + 2)^2}}\,dx$$

$$= \frac{1}{\sqrt{16}}\int \frac{1}{\sqrt{1 - \left(\frac{x + 2}{4}\right)^2}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{1 - \left(\frac{x + 2}{4}\right)^2}}\,dx$$

Using direct reverse, we begin with $y = \sin^{-1}\left(\frac{x + 2}{4}\right)$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x+2}{4}\right)^2}} \times \frac{1}{4}$$

Therefore $\frac{1}{4}\int \frac{1}{\sqrt{1-\left(\frac{x+2}{4}\right)^2}}\,dx = \sin^{-1}\left(\frac{x+2}{4}\right) + c.$

Exercise 2

1 $\int \frac{1}{9+x^2}\,dx$

2 $\int \frac{1}{\sqrt{25-x^2}}\,dx$

3 $\int \frac{-1}{\sqrt{36-x^2}}\,dx$

4 $\int \frac{1}{1+\frac{x^2}{9}}\,dx$

5 $\int \frac{1}{\sqrt{2-\frac{x^2}{4}}}\,dx$

6 $\int_1^p \frac{1}{3+x^2}\,dx$

7 $\int_0^1 \frac{1}{49+x^2}\,dx$

8 $\int \frac{1}{\sqrt{1-3x^2}}\,dx$

9 $\int \frac{1}{\sqrt{-x^2+2x}}\,dx$

10 $\int \frac{1}{4x^2+8x+20}\,dx$

11 $\int \frac{1}{x^2+6x+18}\,dx$

12 $\int \frac{-5}{\sqrt{-x^2-4x+5}}\,dx$

13 $\int \frac{1}{9x^2+6x+5}\,dx$

14 $\int \frac{1}{x^2+3x+4}\,dx$

15 $\int \frac{4}{\sqrt{-x^2+3x+10}}\,dx$

16 $\int \frac{1}{\sqrt{-9x^2+18x+99}}\,dx$

17 $\int_0^{0.5} \frac{1}{x^2+4x+7}\,dx$

18 $\int_{-2}^{p} \frac{1}{\sqrt{-x^2-6x-6}}\,dx$

15.3 Integration of powers of trigonometric functions

To integrate powers of trigonometric functions the standard results and methods of direct reverse are again used, but trigonometric identities are also required.

Even powers of sine and cosine

For these we use the double angle identity $\cos 2x = \cos^2 x - \sin^2 x$.

Example

$\int \sin^2 x\,dx$

Knowing that $\cos 2x = \cos^2 x - \sin^2 x$

$\Rightarrow \cos 2x = 1 - 2\sin^2 x$

$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

$\Rightarrow \int \sin^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\int (1-\cos 2x)\,dx$

$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) = \frac{1}{4}(2x - \sin 2x) + c$

We cannot use this idea with $\sin^3 x$. If $y = \sin^3 x$, then $\frac{dy}{dx} = 3\sin^2 x \cos x$ and so there is a cosine term that creates a problem. This illustrates a major difference between differentiation and integration.

Example

$\int \cos^4 x\,dx$

Knowing that

$\cos 2x = \cos^2 x - \sin^2 x$

$\Rightarrow \cos 2x = 2\cos^2 x - 1$

$\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$

$\Rightarrow \int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \int \left(\frac{1 + 2\cos 2x + \cos^2 2x}{4}\right)dx$

$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$

Using the double angle formula again on $\cos^2 2x$.

$\cos 4x = \cos^2 2x - \sin^2 2x$

$\Rightarrow \cos 4x = 2\cos^2 2x - 1$

$\Rightarrow \cos^2 2x = \frac{1+\cos 4x}{2}$

$\Rightarrow \int \cos^4 x\,dx = \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2}\right)dx$

$= \frac{1}{8}\int (2 + 4\cos 2x + 1 + \cos 4x)\,dx$

$= \frac{1}{8}\int (3 + 4\cos 2x + \cos 4x)\,dx$

$= \frac{1}{8}\left(3x + \frac{4\sin 2x}{2} + \frac{\sin 4x}{4}\right)$

$= \frac{1}{32}(12x + 8\sin 2x + \sin 4x) + c$

For higher even powers, it is a matter of repeating the process as many times as necessary. This can be made into a general formula, but it is beyond the scope of this curriculum.

If integration of even powers of multiple angles is required the same method can be used.

Example

$\int \sin^2 8x \, dx$

This time $\cos 16x = \cos^2 8x - \sin^2 8x$

$\Rightarrow \cos 16x = 1 - 2\sin^2 8x$

$\Rightarrow \sin^2 8x = \dfrac{1 - \cos 16x}{2}$

$$\int \sin^2 8x \, dx = \int \left(\frac{1 - \cos 16x}{2} \right) dx$$

$$= \frac{x}{2} - \frac{\sin 16x}{32} = \frac{1}{32}(16x - \sin 16x) + c$$

Odd powers of sine and cosine

For these use the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ with the aim of leaving a single power of sine multiplied by a higher power of cosine or a single power of cosine multiplied by a higher power of sine.

Example

$\int \sin^3 x dx$

$= \int \sin x \sin^2 x dx$

$= \int \sin x(1 - \cos^2 x) dx$ using the identity $\cos^2 x + \sin^2 x = 1$

$= \int (\sin x - \cos^2 x \sin x) dx$

To find $\int \cos^2 x \sin x dx$ the method of direct reverse is used.

This begins with $y = \cos^3 x = (\cos x)^3$

$$\Rightarrow \frac{dy}{dx} = -3(\cos x)^2 \sin x = -3\cos^2 x \sin x$$

$$\Rightarrow -3\int \cos^2 x \sin x dx = \cos^3 x$$

$$\Rightarrow \int \cos^2 x \sin x dx = -\frac{1}{3}\cos^3 x + k$$

$$\int (\sin x - \cos^2 x \sin x) dx = -\cos x + \frac{1}{3}\cos^3 x + c$$

Unlike even powers of cosine and sine, this is a one-stage process, no matter how high the powers become. This is demonstrated in the next example.

Example

$\int \cos^7 x dx$

$= \int \cos x \cos^6 x dx$

$= \int \cos x(1 - \sin^2 x)^3 dx$

$= \int \cos x(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) dx$

$= \int (\cos x - 3\sin^2 x \cos x + 3\sin^4 x \cos x - \sin^6 x \cos x) dx$

These can all be integrated using the method of direct reverse.

$\int 3\sin^2 x \cos x dx$ begins with $y = \sin^3 x$

$$\Rightarrow \frac{dy}{dx} = 3\sin^2 x \cos x$$

$$\int 3\sin^2 x \cos x dx = \sin^3 x + k_1$$

$\int 3\sin^4 x \cos x dx$ begins with $y = \sin^5 x$

$$\Rightarrow \frac{dy}{dx} = 5\sin^4 x \cos x$$

$$\Rightarrow 5\int \sin^4 x \cos x dx = \sin^5 x$$

$$\Rightarrow 3\int \sin^4 x \cos x dx = \frac{3}{5}\sin^5 x + k_2$$

$\int \sin^6 x \cos x dx$ begins with $y = \sin^7 x$

$$\Rightarrow \frac{dy}{dx} = 7\sin^6 x \cos x$$

$$\Rightarrow 7\int \sin^6 x \cos x dx = \sin^7 x$$

$$\Rightarrow \int \sin^6 x \cos x dx = \frac{1}{7}\sin^7 x + k_3$$

Hence $\int (\cos x - 3\sin^2 x \cos x + 3\sin^4 x \cos x - \sin^6 x \cos x) dx$

$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$, where $c = k_1 + k_2 + k_3$.

Integrating odd powers of multiple angles works in the same way.

Example

$\int \cos^3 4x\,dx$

$= \int \cos 4x \cos^2 4x\,dx$

$= \int \cos 4x(1 - \sin^2 4x)\,dx$

$= \int (\cos 4x - \sin^2 4x \cos 4x)\,dx$

To find $\int \sin^2 4x \cos 4x\,dx$ the method of direct reverse is used.

This begins with $y = \sin^3 4x$

$\Rightarrow \frac{dy}{dx} = 3\sin^2 4x \cos 4x \times 4 = 12\sin^2 4x \cos 4x$

$\Rightarrow 12\int \sin^2 4x \cos 4x\,dx = \sin^3 4x$

$\Rightarrow \int \sin^2 4x \cos 4x\,dx = \frac{1}{12}\sin^3 4x + k$

$\Rightarrow \int (\cos 4x - \sin^2 4x \cos 4x)\,dx = \frac{\sin 4x}{4} - \frac{1}{12}\sin^3 4x + c$

Often this technique will work with mixed powers of sine and cosine and the aim is still to leave a single power of sine multiplied by a higher power of cosine or a single power of cosine multiplied by a higher power of sine.

Example

$\int \sin^3 x \cos^2 x\,dx$

Since it is sine that has the odd power, this is the one that is split.

$\int \sin x \sin^2 x \cos^2 x\,dx = \int \sin x(1 - \cos^2 x)\cos^2 x\,dx$

$= \int (\cos^2 x \sin x - \cos^4 x \sin x)\,dx$

Now these can both be integrated using the method of direct reverse.

$\int \cos^2 x \sin x\,dx$ begins with $y = \cos^3 x$

$\Rightarrow \frac{dy}{dx} = -3\cos^2 x \sin x$

$\Rightarrow -3\int \cos^2 x \sin x\,dx = \cos^3 x$

$\int \cos^2 x \sin x\,dx = -\frac{1}{3}\cos^3 x + k_1$

$\int \cos^4 x \sin x\,dx$ begins with $y = \cos^5 x$

$\Rightarrow \frac{dy}{dx} = -5\cos^4 x \sin x$

$\Rightarrow -5\int \cos^4 x \sin x\,dx = \cos^5 x$

$\Rightarrow \int \cos^4 x \sin x\,dx = -\frac{1}{5}\cos^5 x + k_2$

Therefore $\int (\cos^2 x \sin x - \cos^4 x \sin x)\,dx = -\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$, where $c = k_1 + k_2$.

Powers of tan x

In this case the identity $1 + \tan^2 x = \sec^2 x$ is used with the aim of getting $\tan x$, $\sec^2 x$ or a power of tan x multiplied by $\sec^2 x$. It should also be remembered that $\int \tan x\,dx = -\ln|\cos x| + c$ and $\int \sec^2 x\,dx = \tan x + c$.

Example

$\int \tan^3 x\,dx$

This is first turned into $\int \tan x \tan^2 x\,dx$.

We need to extract $\tan^2 x$ out of the power of tan x in order to produce $\sec^2 x - 1$

Using the identity gives $\int \tan x(\sec^2 x - 1)\,dx = \int \tan x \sec^2 x - \tan x\,dx$

To find $\int \tan x \sec^2 x\,dx$

direct reverse is used. This happens because the derivative of tan x is $\sec^2 x$ and also explains why it is necessary to have $\sec^2 x$ with the power of tan x.
To integrate this we begin with $y = \tan^2 x$.

So $\frac{dy}{dx} = 2\tan x \sec^2 x$

Hence $2\int \tan x \sec^2 x\,dx = \tan^2 x$,

Therefore $\int \tan x \sec^2 x\,dx = \frac{1}{2}\tan^2 x + k$

Thus $\int \tan^3 x\,dx = \int \tan x \sec^2 x - \tan x\,dx$

$= \frac{1}{2}\tan^2 x + \ln|\cos x| + c$

Example

$$\int \tan^5 x dx$$

First change this to $\int \tan^3 x \tan^2 x dx$.

Using the identity: $\int \tan^3 x(\sec^2 x - 1)dx = \int (\tan^3 x \sec^2 x - \tan^3 x)dx$

The integral of $\tan^3 x$ was done in the example above and the result will just be quoted here. The integral of $\tan^3 x \sec^2 x$ is done by direct reverse.

This begins with $y = \tan^4 x$

$$\Rightarrow \frac{dy}{dx} = 4\tan^3 x \sec^2 x$$

$$\Rightarrow 4\int \tan^3 x \sec^2 x dx = \tan^4 x$$

$$\Rightarrow \int \tan^3 x \sec^2 x dx = \frac{1}{4}\tan^4 x + k$$

Hence $\int \tan^5 x dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln \cos x + c$.

If multiple angles are used, this does not change the method.

As with even and odd powers of sine and cosine, as the powers get higher, we are just repeating earlier techniques and again this could be generalized.

Example

$$\int \tan^2 2x dx$$

This time the identity $\tan^2 2x = \sec^2 2x - 1$ is used.

$$\int \tan^2 2x dx = \int (\sec^2 2x - 1)dx$$

$$= \frac{1}{2}\tan 2x - x + c$$

Exercise 3

1 $\int \cos^3 x dx$

2 $\int \sin^3 2x dx$

3 $\int \sin^5 x dx$

4 $\int_0^p \tan^2 2x dx$

5 $\int \cos^2 2x dx$

6 $\int \sin^2 2x dx$

7 $\int \sin^4 x dx$

8 $\int \sin^9 x dx$

9 $\int \tan^3 3x dx$

10 $\int \sin^2 x \cos^2 x dx$

11 $\int \tan^3 x \sec^4 x dx$

12 $\int \sin^3 2x \cos^2 2x dx$

13 $\int_0^p \frac{\sin^2 x}{\sec^3 x} dx$

15.4 Selecting the correct technique 1

The skill in integration is often to recognize which techniques to apply. Exercise 4 contains a mixture of questions.

Exercise 4

1 $\int (x + 2)^4 dx$

2 $\int (2 + 7x)^3 dx$

3 $\int \frac{1}{\sqrt{1 - 2x}} dx$

4 $\int \left(\frac{3}{(2x + 1)^3} + \sqrt{1 + 2x}\right) dx$

5 $\frac{1}{4}\int \sqrt{3 + 5x}\, dx$

6 $\int \left(\sqrt{1 - x} + \frac{1}{\sqrt{1 - x}} - \frac{1}{(1 - x)^2}\right) dx$

7 $3\int \cos\left(4x - \frac{\pi}{2}\right) dx$

8 $\int \frac{2}{1 + 4x^2} dx$

9 $\int \frac{\sin x}{3 - 4\cos x} dx$

10 $\int \sec^2\left(\frac{\pi}{3} - 2x\right) dx$

11 $2\int \sin(3x + \alpha)\, dx$

12 $\int e^{4x+1} dx$

13 $\int 2^x dx$

14 $\int \frac{1}{3x + 1} dx$

15 $\int \frac{2x}{x^2 + 4} dx$

16 $\int \frac{x + 1}{x^2 + 2x + 3} dx$

17 $\int \frac{x^3}{x^4 + 3} dx$

18 $\int \frac{3}{\sqrt{1 - 9x^2}} dx$

19 $\int \frac{6}{4 + 16x^2} dx$

20 $\int (x + 3)(x^2 + 6x - 8)^6 dx$

21 $\int \cos 2x(\sin 2x + 3)^4 dx$

22 $\int \mathrm{cosec}^2\frac{x}{2} e^{1 - \cot\frac{x}{2}} dx$

23 $\int \sqrt{x}(1 + x^{\frac{3}{2}})^7 dx$

24 $\int \cos^3 x \sin x\, dx$

25 $\int \frac{\mathrm{cosec}^2 x}{(\cot x - 3)^3} dx$

26 $\int \frac{e^x}{e^x + 2} dx$

27 $\int \frac{e^x}{(e^x + 2)^{\frac{1}{2}}} dx$

28 $\int \sin^4 2x\, dx$

29 $\int \frac{x + 1}{x^2 + 2x + 3} dx$

30 $\int \frac{2}{x^2 + 2x + 3} dx$

31 $\int \frac{2}{\sqrt{-x^2 + 4x + 5}} dx$

32 $\int \frac{-2x + 4}{\sqrt{-x^2 + 4x + 5}} dx$

33 $\int \sin 2x(\sin^2 x + 3)^4 dx$

15.5 Integration by substitution

The method we have called direct reverse is actually the same as substitution except that we do the substitution mentally. The questions we have met so far could all have been done using a method of substitution, but it is much more time consuming. However, certain more complicated questions require a substitution to be used. If a question requires substitution then this will often be indicated, as will the necessary substitution. Substitution is quite straightforward, apart from "dealing with the dx part". Below is a proof of the equivalence of operators, which will allow us to "deal with dx".

Proof

Consider a function of u, f(u).

$$\frac{d}{dx}[f(u)] = \frac{du}{dx} \times f'(u)$$

Hence integrating both sides gives $\int \frac{du}{dx}f'(u)dx = f(u) + k$ (equation 1).

Also $f'(u) = \frac{d}{du}f(u)$.

Therefore $\int f'(u)du = f(u) + k$ (equation 2).

Combining equation 1 with equation 2 gives $\int \frac{du}{dx}f'(u)dx = \int f'(u)du$

Therefore $\int \ldots \frac{du}{dx}\,dx = \int \ldots du$ where $\ldots$ is the function being integrated.

This is known as the equivalence of operators.

The question is how to use it. There is a great temptation to treat "dx" as part of a fraction. In the strictest sense it is not, it is a piece of notation, but at this level of mathematics most people do treat it as a fraction and in the examples we will do so. The equivalence of operators shown above demonstrates that treating $\frac{dy}{dx}$ as a fraction will also work.

Example

Find $\int \cos x(1 + \sin x)^{\frac{1}{2}}\,dx$ using the substitution $u = 1 + \sin x$.

This example could also be done by direct reverse. It is possible that an examination could ask for a question to be done by substitution when direct reverse would also work.

$$(1 + \sin x)^{\frac{1}{2}} = u^{\frac{1}{2}}$$

$$\frac{du}{dx} = \cos x$$

$$\Rightarrow \cos x\,dx = du$$

This is the same as using the equivalence of operators, which would work as follows:

$$\int \ldots \frac{du}{dx}\,dx = \int \ldots du$$

$$\Rightarrow \int \ldots \cos x\,dx = \int \ldots du$$

Making the substitution gives $\int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}$

The answer cannot be left in this form and we need to substitute for x.

So $\int \cos x(1 + \sin x)^{\frac{1}{2}}\,dx = \frac{2}{3}(1 + \sin x)^{\frac{3}{2}} + c$

Example

Find $\int 3x\sqrt{4x - 1}\,dx$ using the substitution $u = 4x - 1$.

$$\sqrt{4x - 1} = u^{\frac{1}{2}}$$

$$3x = 3\left(\frac{u + 1}{4}\right)$$

$$\frac{du}{dx} = 4$$

Hence $dx = \frac{du}{4}$

$$\Rightarrow \int 3x\sqrt{4x - 1}\,dx = \int 3\left(\frac{u + 1}{4}\right)u^{\frac{1}{2}}\frac{du}{4}$$

$$= \frac{3}{16}\int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du$$

$$= \frac{3}{16}\left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right] = \frac{u^{\frac{3}{2}}}{40}[3u + 5]$$

Hence $\int 3x\sqrt{4x - 1}\,dx = \frac{(4x - 1)^{\frac{3}{2}}}{40}[3(4x - 1) + 5]$

$$= \frac{(4x - 1)^{\frac{3}{2}}}{40}(12x + 2)$$

$$= \frac{(4x - 1)^{\frac{3}{2}}}{20}(6x + 1) + c$$

Definite integration works the same way as with other integration, but the limits in the substitution need to be changed.

Example

Find $\int_0^p (x + 1)(2x - 1)^9\,dx$ using the substitution $u = 2x - 1$.

$$(2x - 1)^9 = u^9$$

$$x + 1 = \frac{u + 3}{2}$$

If the question has two numerical limits and appears on a calculator paper, then perform the calculation directly on a calculator.

$$\frac{du}{dx} = 2$$

$$\Rightarrow dx = \frac{du}{2}$$

Because this is a question of definite integration, the limits must be changed. The reason for this is that the original limits are values of x and we now need values of u as we are integrating with respect to u.

When $x = 0, u = -1$

When $x = p, u = 2p - 1$

Hence the integral now becomes

$$\int_{-1}^{2p-1} \left(\frac{u+3}{2}\right)u^9 \frac{du}{2} = \frac{1}{4}\int_{-1}^{2p-1} (u^{10} + 3u^9)du$$

$$= \frac{1}{4}\left[\frac{u^{11}}{11} + \frac{3u^{10}}{10}\right]_{-1}^{2p-1}$$

$$= \frac{1}{4}\left[\left(\frac{(2p-1)^{11}}{11} + \frac{3(2p-1)^{10}}{10}\right) - \left(-\frac{1}{11} + \frac{3}{10}\right)\right]$$

$$= \frac{1}{440}[10(2p-1)^{11} + 33(2p-1)^{10} - 23]$$

When limits are changed using substitution, it is sometimes the case that the limits switch around and the lower limit is bigger than the upper limit.

Example

Evaluate $\displaystyle\int_{-1}^{1} \frac{(x+1)dx}{(2-x)^4}$.

On a calculator paper this would be done directly by calculator.

$$\Rightarrow \int_{-1}^{1} \frac{(x+1)dx}{(2-x)^4} = 0.519$$

On a non-calculator paper we would proceed as follows.

Let $u = 2 - x$

$(2 - x)^4 = u^4$

$x + 1 = 3 - u$

$$\frac{du}{dx} = -1$$

$\Rightarrow dx = -du$

When $x = -1, u = 3$

When $x = 1, u = 1$

Therefore $\displaystyle\int_{-1}^{1} \frac{(x+1)dx}{(2-x)^4}$ becomes

$$\int_{3}^{1} (u^{-3} - 3u^{-4})du = \left[\frac{u^{-2}}{-2} - \frac{3u^{-3}}{-3}\right]_3^1$$

$$= \left[\frac{-1}{2u^2} + \frac{1}{u^3}\right]_3^1$$

$$= \left(\frac{-1}{2} + 1\right) - \left(\frac{-1}{18} + \frac{1}{27}\right)$$

$$= \frac{14}{27}$$

The substitutions dealt with so far are fairly intuitive, but some of them are less obvious. In this case the question will sometimes state the substitution.

Example

Find $\int \sqrt{1-x^2}\,dx$ using the substitution $x = \sin\theta$.

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$$

$$\frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta\, d\theta$$

Hence $\int \sqrt{1-x^2}\,dx$ becomes $\int \cos^2\theta\, d\theta$.

Now $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\Rightarrow \cos 2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\Rightarrow \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2} d\theta$$

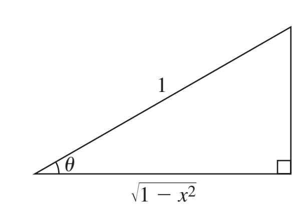
$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4}$$

We now substitute back for x.

Given that $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

Also $\sin 2\theta = 2\sin\theta\cos\theta$

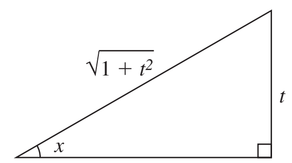
From the triangle below, $\cos\theta = \sqrt{1-x^2}$.



Hence $\displaystyle\int \sqrt{1-x^2}dx = \frac{\sin^{-1} x}{2} + 2x\sqrt{1-x^2} + c.$

Example

Find $\displaystyle\int \frac{2\tan x}{\cos 2x}\,dx$ using the substitution $t = \tan x$.



From the triangle above, $\sin x = \dfrac{t}{\sqrt{1+t^2}}$ and $\cos x = \dfrac{1}{\sqrt{1+t^2}}$.

Hence $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{1}{1+t^2} - \dfrac{t^2}{1+t^2} = \dfrac{1-t^2}{1+t^2}$.

If $t = \tan x$, then $\dfrac{dt}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + t^2$.

$$\Rightarrow dx = \frac{dt}{1+t^2}$$

$$\int \frac{2\tan x}{\cos 2x}\,dx = \int \frac{2t}{\dfrac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2} = \int \frac{2t}{1-t^2}\,dt$$

This is now done by direct reverse and begins with $y = \ln|1 - t^2|$.

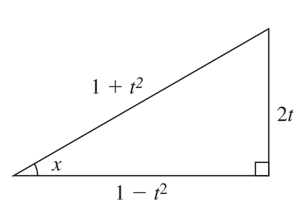
$$\Rightarrow \frac{dy}{dt} = \frac{2t}{1-t^2}$$

Therefore $\displaystyle\int \frac{2\tan x}{\cos 2x}\,dx = \int \frac{2t}{1-t^2}\,dt = \ln|1-t^2| = \ln|1-\tan^2 x| + c.$

Example

Find $\displaystyle\int \frac{1}{4+\sin x}\,dx$ using the substitution $t = \tan\dfrac{x}{2}$.

Since $\tan x = \dfrac{2\tan\dfrac{x}{2}}{1-\tan^2\dfrac{x}{2}}$



From the above diagram, $\sin x = \dfrac{2t}{1+t^2}$.

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) = \frac{1}{2}(1+t^2)$$

$$\Rightarrow dx = \frac{2dt}{1+t^2}$$

Therefore $\displaystyle\int \frac{1}{4+\sin x}\,dx = \int \frac{2\dfrac{dt}{1+t^2}}{4 + \dfrac{2t}{1+t^2}} = \int \frac{2}{4(1+t^2)+2t}\,dt$

$$= \int \frac{2}{4\left(t^2 + \dfrac{t}{2} + 1\right)}\,dt = \int \frac{2}{4\left[\left(t+\dfrac{1}{4}\right)^2 + \dfrac{15}{16}\right]}\,dt$$

$$= \frac{16}{15}\int \frac{1}{1 + \left[\dfrac{2}{\sqrt{15}}\left(2t + \dfrac{1}{2}\right)\right]^2}\,dt$$

This is now integrated by direct reverse beginning with $y = \tan^{-1}\dfrac{2}{\sqrt{15}}\left(2t + \dfrac{1}{2}\right)$

$$\Rightarrow \frac{dy}{dt} = \frac{\dfrac{2}{\sqrt{15}}}{1 + \left[\dfrac{2}{\sqrt{15}}\left(2t + \dfrac{1}{2}\right)\right]^2}$$

$$\Rightarrow \frac{2}{\sqrt{15}}\int \frac{1}{1 + \left[\dfrac{2}{\sqrt{15}}\left(2t + \dfrac{1}{2}\right)\right]^2}\,dt = \tan^{-1}\frac{2}{\sqrt{15}}\left(2t + \frac{1}{2}\right)$$

$$\Rightarrow \frac{16}{15}\int \frac{1}{1 + \left[\dfrac{2}{\sqrt{15}}\left(2t + \dfrac{1}{2}\right)\right]^2}\,dt = \frac{8}{\sqrt{15}}\tan^{-1}\frac{2}{\sqrt{15}}\left(2t + \frac{1}{2}\right)$$

$$\Rightarrow \int \frac{1}{4+\sin x}\,dx = \frac{8}{\sqrt{15}}\tan^{-1}\frac{1}{\sqrt{15}}\left(4\tan\frac{x}{2} + 1\right) + c$$

Exercise 5

1 Find $\displaystyle\int x(x^2+3)^5\,dx$ using the substitution $u = x^2 + 3$.

2 Find $\displaystyle\int \frac{3x+1}{6x^2+4x-13}\,dx$ using the substitution $u = 6x^2 + 4x - 13$.

3 Find $\displaystyle\int \frac{\cos 2x}{\sqrt{1-\sin 2x}}\,dx$.

4 Find $\displaystyle\int x\sqrt{x-2}\,dx$ using the substitution $u = x - 2$.

5 Find $\displaystyle\int_1^p \frac{x}{\sqrt{2x-1}}\,dx$.

6 Find $\displaystyle\int \frac{(x+3)}{\sqrt{2x+1}}\,dx$ using the substitution $u = 2x + 1$.

7 Find $\displaystyle\int \frac{2x}{1+x^4}\,dx$ using the substitution $u = x^2$.

8 Find $\displaystyle\int (x+2)\sqrt{3x-4}\,dx$.

9 Find $\displaystyle\int_1^p (2x-1)(x-2)^3\,dx$ using the substitution $u = x - 2$.

10 Find $\displaystyle\int \frac{x}{(2x-1)^4}\,dx$ using the substitution $u = 2x - 1$.

11 Find $\int \frac{2x+1}{(x-3)^6}\,dx$. **12** Find $\int_9^p \frac{x}{\sqrt{x-2}}\,dx$. **13** Find $\int \frac{x^3}{(x+5)^2}\,dx$.

14 Find $\int_2^p x\sqrt{5x+2}\,dx$. **15** Find $\int \frac{x(x-4)}{(x-2)^2}\,dx$.

16 Find $\int \frac{1}{\cos^2 x + 4\sin^2 x}\,dx$ using the substitution $t = \tan x$.

17 Find $\int \sqrt{9-9x^2}\,dx$ using the substitution $x = \sin\theta$.

18 Find $\int \frac{1}{3-5\cos x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.

19 Find $\int_{0.5}^{p} \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$.

20 Find $\int \frac{1}{5\sin^2 x + \cos^2 x}\,dx$ using the substitution $t = \tan x$.

21 Find $\int_0^p \frac{6}{5+3\sin x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.

22 Find $\int \frac{1}{8+8\cos 4x}\,dx$ using the substitution $t = \tan 2x$.

23 Find $\int \frac{4}{3x\sqrt{x^n-1}}\,dx$ using the substitution $u^2 = x^n - 1$.

15.6 Integration by parts

As was mentioned earlier in the chapter, not all products can be integrated by the method of direct reverse. Integration by parts is another technique and tends to be used when one half of the product is not related to the other half. Direct reverse is basically undoing the chain rule and integration by parts is basically reversing the product rule. However, unlike direct reverse, this does not mean that it is used for those answers that came from the product rule.

We will begin by showing the formula.

We know that

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \quad \text{where } u \text{ and } v \text{ are both functions of } x.$$

$$\Rightarrow v\frac{du}{dx} = \frac{d}{dx}(uv) - u\frac{dv}{dx}$$

Integrating both sides with respect to x gives:

$$\int v\frac{du}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int u\frac{dv}{dx}\,dx$$

Now $\int \frac{d}{dx}(uv)\,dx$ is just uv, so:

$$\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$$

This is the formula for integration by parts.

The basic method is as follows.

Let one part of the product be v and one part $\frac{du}{dx}$. Calculate u and $\frac{dv}{dx}$ and then use the formula.

Unlike the product rule in differentiation, in some cases it makes a difference which part is v and which part is $\frac{du}{dx}$ and in other cases it makes no difference. The choice depends on what can be integrated, and the aim is to make the problem easier. The table below will help.

One half of product	Other half of product	Which do you differentiate?
Power of x	Trigonometric ratio	Power of x
Power of x	Inverse trigonometric ratio	Power of x
Power of x	Power of e	Power of x
Power of x	ln f(x)	ln f(x)
Power of e	sin f(x), cos f(x)	Does not matter

This can also be summarized as a priority list.

Which part is v?
1. Choose ln f(x).
2. Choose the power of x.
3. Choose $e^{f(x)}$ or sin f(x), cos f(x).

Example

Find $\int xe^x\,dx$.

Using the formula $\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$,

let $v = x$ and $\frac{du}{dx} = e^x$

$\Rightarrow \frac{dv}{dx} = 1$ and $u = \int e^x\,dx = e^x$

Now substitute the values in the formula.

$$\int xe^x\,dx = xe^x - \int 1e^x\,dx$$

$$\Rightarrow \int xe^x\,dx = xe^x - e^x + c$$

x is differentiated here since it will differentiate to 1 and allow the final integration to be carried out.

It is possible to leave out the mechanics of the question once you feel more confident about the technique.

Example

Find $\int x \sin x dx$.

$$\int x \sin x dx = -x \cos x - \int -\cos x \times 1 dx$$

It is not recommended to try to simplify the signs and constants at the same time as doing the integration!

$$= -x \cos x + \int \cos x dx$$

$$= -x \cos x + \sin x + c$$

Example

Find $\int x^3 \ln x dx$.

Using the formula $\int v\frac{du}{dx} dx = uv - \int u\frac{dv}{dx} dx$,

let $v = \ln x$ and $\frac{du}{dx} = x^3$

There is no choice but to differentiate $\ln x$ since it cannot be integrated at this point.

$\Rightarrow \frac{dv}{dx} = \frac{1}{x}$ and $u = \int x^3 dx = \frac{x^4}{4}$

Substituting the values in the formula gives:

$$\int x^3 \ln x dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \times \frac{1}{x} dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$$

Therefore $\int x^3 \ln x dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} = \frac{x^4}{16}(4 \ln x - 1) + c$

Example

Find $\int_0^p 4x \cos x dx$.

Using the formula $\int v\frac{du}{dx} dx = uv - \int u\frac{dv}{dx} dx$,

let $v = 4x$ and $\frac{du}{dx} = \cos x$

$\Rightarrow \frac{dv}{dx} = 4$ and $u = \int \cos x dx = \sin x$

Substituting the values in the formula gives:

$$\int_0^p 4x \cos x dx = [4x \sin x]_0^p - \int_0^p 4 \sin x dx$$

$$= [4p \sin p - 0] - [-4 \cos x]_0^p$$

$$= [4p \sin p - 0] + [4 \cos p + 4]$$

$$= 4p \sin p + 4 \cos p - 4$$

Example

Find $\int x^2 e^x dx$.

Here the integration by parts formula will need to be applied twice.

Using the formula $\int v\frac{du}{dx} dx = uv - \int u\frac{dv}{dx} dx$,

let $v = x^2$ and $\frac{du}{dx} = e^x$

$\Rightarrow \frac{dv}{dx} = 2x$ and $u = \int e^x dx = e^x$

$\Rightarrow \int xe^x dx = x^2e^x - \int 2xe^x dx = x^2e^x - 2\int xe^x dx$

It is always a good idea to take the constants outside the integral sign.

We need to find $\int xe^x dx$. This is again done using the method of integration by parts.

Using the formula $\int v\frac{du}{dx} dx = uv - \int u\frac{dv}{dx} dx$,

let $v = x$ and $\frac{du}{dx} = e^x$

$\Rightarrow \frac{dv}{dx} = 1$ and $u = \int e^x dx = e^x$

$\Rightarrow \int x e^x dx = xe^x - \int 1e^x dx = xe^x - e^x$

Combining the two gives:

$$\int xe^x dx = x^2e^x - 2(xe^x - e^x)$$

$$= x^2e^x - 2xe^x - 2e^x + c$$

Example

Find $\int e^{2x} \sin x dx$.

This is a slightly different case, since it makes no difference which part is integrated and which part is differentiated. With a little thought this should be obvious since, excluding constants, repeated integration or differentiation of these functions gives the same pattern of answers. Remember the aid

$$\text{Differentiate} \downarrow \begin{array}{c} S \\ C \\ -S \\ -C \end{array} \uparrow \text{Integrate}$$

and the fact that functions of e^x differentiate or integrate to themselves.

We begin by letting $\int e^{2x} \sin x\,dx = I$.

Using the formula $\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$,

let $v = e^{2x}$ and $\frac{du}{dx} = \sin x$

$\Rightarrow \frac{dv}{dx} = 2e^{2x}$ and $u = \int \sin x\,dx = -\cos x$

$$\Rightarrow I = \int e^{2x} \sin x\,dx = -e^{2x}\cos x - \int -\cos x \times 2e^{2x}\,dx$$

$$= -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$

Applying the formula again, being very careful to ensure that we continue to integrate the trigonometric function and differentiate the power of e gives:

$v = e^{2x}$ and $\frac{du}{dx} = \cos x$

$\Rightarrow \frac{dv}{dx} = 2e^{2x}$ and $u = \int \cos x\,dx = \sin x$

$$\text{Hence } \int e^{2x}\cos x\,dx = e^{2x}\sin x - \int \sin x \times 2e^{2x}\,dx$$

$$= e^{2x}\sin x - 2\int e^{2x}\sin x\,dx$$

This is the original integral I.

$$\text{Hence } I = \int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right)$$

$$\Rightarrow I = -e^{2x}\cos x + 2e^{2x}\sin x - 4I$$

$$\Rightarrow 5I = -e^{2x}\cos x + 2e^{2x}\sin x$$

Calling the original integral I makes this rearrangement easier.

$$\Rightarrow I = \frac{1}{5}(-e^{2x}\cos x + 2e^{2x}\sin x) = \frac{e^{2x}}{5}(-\cos x + 2\sin x) + c$$

Example

Find $\int \ln x\,dx$.

This is done as a special case of integration by parts. However, it is not a product of two functions. To resolve this issue we let the other function be 1.

Hence this becomes $\int 1 \ln x\,dx$ and the integration by parts formula is applied as usual.

Using the formula $\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$,

let $v = \ln x$ and $\frac{du}{dx} = 1$

$\Rightarrow \frac{dv}{dx} = \frac{1}{x}$ and $u = \int 1\,dx = x$

$$\text{Hence } \int 1 \ln x\,dx = x\ln x - \int x \times \frac{1}{x}\,dx = x\ln x - \int 1\,dx = x\ln x - x + c$$

To integrate inverse trigonometric functions an identical method is used, for example $\int \cos^{-1} x\,dx = \int 1 \cdot \cos^{-1} x\,dx$.

Exercise 6

Find these integrals using the method of integration by parts.

1 $\int x\cos x\,dx$ 2 $\int x e^{2x}\,dx$ 3 $\int x^4 \ln x\,dx$

4 $\int x\sin 2x\,dx$ 5 $\int_1^p x(x+1)^9\,dx$ 6 $\int x^2\sin x\,dx$

7 $\int x^2 e^{2x}\,dx$ 8 $\int x^2\ln 3x\,dx$ 9 $\int 3x^2\ln 8x\,dx$

10 $\int x^2 e^{-3x}\,dx$ 11 $\int e^x\cos x\,dx$ 12 $\int \sin^{-1} x\,dx$

13 $\int \tan^{-1} x\,dx$ 14 $\int e^{2x}(2x-1)\,dx$ 15 $\int e^{3x}\cos x\,dx$

16 $\int e^{2x}\sin 3x\,dx$ 17 $\int 2e^x \sin x\cos x\,dx$ 18 $\int x^n\ln x\,dx$

19 $\int e^{ax}\sin bx\,dx$ 20 $\int_0^p x(2x+1)^n\,dx$

15.7 Miscellaneous techniques

There are two other techniques that need to be examined. These methods are normally only used when it is suggested by a question or when earlier techniques do not work.

Splitting the numerator

This is a trick that can really help when the numerator is made up of two terms. Often these questions cannot be tackled by a method of direct reverse as the derivative of the denominator does not give a factor of the numerator. Substitution is unlikely to simplify the situation and integration by parts does not produce an integral that is any simpler.

Example

Find $\int \frac{2x+1}{x^2+1}\,dx$.

Splitting the numerator gives two integrals.

$$\int \frac{2x+1}{x^2+1}\,dx = 2\int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

The first integral can be done by direct reverse and the second one is a standard result. To integrate the first integral we begin with $y = \ln|x^2 + 1|$.

So $\frac{dy}{dx} = \frac{2x}{x^2 + 1}$

Hence $2\int \frac{x}{x^2 + 1}\,dx = \ln|x^2 + 1|$

Therefore $\int \frac{2x + 1}{x^2 + 1}\,dx = 2\int \frac{x}{x^2 + 1}\,dx + \int \frac{1}{x^2 + 1}\,dx$

$= \ln|x^2 + 1| + \tan^{-1} x + c$

Example

Find $\int \frac{2x}{x^2 + 2x + 26}\,dx$.

This is a slightly different case as we now make the numerator $2x + 2$ and then split the numerator. Hence

$\int \frac{2x}{x^2 + 2x + 26}\,dx$ becomes $\int \frac{2x + 2}{x^2 + 2x + 26}\,dx - \int \frac{2}{x^2 + 2x + 26}\,dx$.

The first integral is calculated by direct reverse and the second integral will become a function of inverse tan.

Consider the first integral.

$\int \frac{2x + 2}{x^2 + 2x + 26}\,dx$

To integrate this we know it began with something to do with

$y = \ln|x^2 + 2x + 26|$.

$\Rightarrow \frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 26}$

Hence $\int \frac{2x + 2}{x^2 + 2x + 26}\,dx = \ln|x^2 + 2x + 26| + k_1$

Now look at the second integral.

$$\int \frac{2}{x^2 + 2x + 26}\,dx = \int \frac{2}{(x + 1)^2 + 25}\,dx = \int \frac{2}{25 + (x + 1)^2}\,dx$$

Here we complete the square on the denominator to produce an inverse tan result.

$$= \frac{2}{25}\int \frac{1}{1 + \left(\frac{x + 1}{5}\right)^2}\,dx$$

Using direct reverse, we begin with $y = \tan^{-1}\left(\frac{x + 1}{5}\right)$.

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x + 1}{5}\right)^2} \times \frac{1}{5}$$

Therefore $\frac{1}{5}\int \frac{1}{1 + \left(\frac{x + 1}{5}\right)^2}\,dx = \tan^{-1}\left(\frac{x + 1}{5}\right)$

$$\Rightarrow \frac{2}{25}\int \frac{1}{1 + \left(\frac{x + 1}{5}\right)^2}\,dx = \frac{2}{5}\tan^{-1}\left(\frac{x + 1}{5}\right) + k_2$$

Putting the two integrals together gives:

$$\int \frac{2x}{x^2 + 2x + 26}\,dx = \int \frac{2x + 2}{x^2 + 2x + 26}\,dx - \int \frac{2}{x^2 + 2x + 26}\,dx$$

$$= \ln|x^2 + 2x + 26| - \frac{2}{5}\tan^{-1}\left(\frac{x + 1}{5}\right) + c$$

Algebraic division

If the numerator is of higher or equal power to the denominator, then algebraic division may help. Again this only needs to be tried if other methods have failed. In Chapter 8, rational functions (functions of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are both polynomials) were introduced when finding non-vertical asymptotes. Algebraic division was used if degree of $P(x) \geq$ degree of $Q(x)$. In order to integrate these functions exactly the same thing is done.

Example

Find $\int \frac{x + 1}{x + 3}\,dx$.

Algebraically dividing the fraction:

$$\begin{array}{r} 1 \\ x + 1\overline{)x + 3} \\ \underline{x + 1} \\ 2 \end{array}$$

So the question becomes $\int 1 + \frac{2}{x + 1}\,dx$.

$\int 1 + \frac{2}{x + 1}\,dx = x + 2\ln|x + 1| + c$

Example

Find $\displaystyle\int \frac{x^2 + 1}{x + 1}\,dx$.

Algebraically dividing the fraction:

$$\begin{array}{r} x - 1 \\ x + 1 \overline{\smash{)}\,x^2 + 0x + 1} \\ \underline{x^2 + x} \\ -x + 1 \\ \underline{-x - 1} \\ 2 \end{array}$$

So the question becomes $\displaystyle\int (x - 1) + \frac{2}{x + 1}\,dx$.

$$\int (x - 1) + \frac{2}{x + 1}\,dx = \frac{x^2}{2} - x + 2\ln|x + 1| + c$$

Exercise 7

Evaluate these integrals.

1 $\displaystyle\int \frac{x + 1}{\sqrt{1 - x^2}}\,dx$ 2 $\displaystyle\int \frac{3x + 4}{x^2 + 4}\,dx$ 3 $\displaystyle\int \frac{x + 5}{x^2 + 3}\,dx$

4 $\displaystyle\int \frac{4x + 7}{x^2 + 4x + 8}\,dx$ 5 $\displaystyle\int \frac{2x + 3}{x^2 + 4x + 6}\,dx$ 6 $\displaystyle\int \frac{-2x - 5}{\sqrt{-x^2 - 6x - 4}}\,dx$

7 $\displaystyle\int \frac{x - 3}{x + 4}\,dx$ 8 $\displaystyle\int \frac{x^2 + 1}{x + 3}\,dx$

15.8 Further integration practice

All the techniques of integration for this curriculum have now been met. The following exercise examines all the techniques. It should be noted that there is often more than one technique that will work. For example, direct reverse questions can be done by substitution and some substitution questions can be done by parts.

Exercise 8

Find these integrals using the method of direct reverse.

1 $\displaystyle\int (\cos 3x + \sin 2x)\,dx$ 2 $\displaystyle\int (4\sqrt{x} + 4\sqrt{x + 1} - 4(1 - 3x)^3)\,dx$

3 $\displaystyle\int \frac{2x}{3x^2 + 1}\,dx$ 4 $\displaystyle\int \frac{4}{1 + 4x^2}\,dx$

5 $\displaystyle\int \sec^2\left(2x - \frac{\pi}{3}\right)dx$ 6 $\displaystyle\int \frac{\cos x}{\sqrt{1 + \sin x}}\,dx$

7 $\displaystyle\int (4x + 2)e^{x^2 + x - 5}\,dx$ 8 $\displaystyle\int \frac{\sin x}{\cos^n x}\,dx$

9 $\displaystyle\int \frac{2}{4x^2 + 8x + 5}\,dx$ 10 $\displaystyle\int \frac{2}{\sqrt{-x^2 - 8x + 9}}\,dx$

Find these by using a substitution.

11 $\displaystyle\int 2x(1 - x)^7\,dx$ using the substitution $u = 1 - x$

12 $\displaystyle\int \frac{x^2}{(x + 5)^2}\,dx$ using the substitution $u = x + 5$

13 $\displaystyle\int 2x\sqrt{3x - 4}\,dx$

14 $\displaystyle\int_1^p \frac{2x + 1}{(x - 3)^6}\,dx$

15 $\displaystyle\int \frac{1}{e^x + e^{-x}}\,dx$ using the substitution $u = e^x$

16 $\displaystyle\int \sqrt{4 - 4x^2}\,dx$ using the substitution $x = \sin\theta$

17 $\displaystyle\int \frac{1}{2 + \cos x}\,dx$ using the substitution $t = \tan\frac{x}{2}$

18 $\displaystyle\int \frac{1}{x^2\sqrt{4 - x^2}}\,dx$ using the substitution $x = 2\sin\theta$

19 $\displaystyle\int \frac{3x}{1 + x^4}\,dx$ using the substitution $x^2 = p$

20 $\displaystyle\int \frac{x + 1}{x\sqrt{x - 2}}\,dx$ using the substitution $x - 2 = p^2$

Find these by integrating by parts.

21 $\displaystyle\int \frac{1}{x^2}\ln x\,dx$ 22 $\displaystyle\int x\,e^{3x}\,dx$ 23 $\displaystyle\int x\cos\left(x + \frac{\pi}{6}\right)dx$

24 $\displaystyle\int e^{-2x}\cos 2x\,dx$ 25 $\displaystyle\int x^2\sin\frac{x}{2}\,dx$ 26 $\displaystyle\int \ln(2x + 1)\,dx$

27 $\displaystyle\int \tan^{-1}\left(\frac{1}{x}\right)dx$ 28 $\displaystyle\int e^{ax}\sin 2x\,dx$

Integrate these trigonometric powers.

29 $\displaystyle\int \cos x\sin^2 x\,dx$ 30 $\displaystyle\int \frac{\cos^2 x}{\operatorname{cosec} x}\,dx$ 31 $\displaystyle\int \frac{\tan^3 x}{\cos^2 x}\,dx$

32 $\displaystyle\int \cos^2\theta\,d\theta$ 33 $\displaystyle\int \sin^2 3x\,dx$ 34 $\displaystyle\int \cos^3 2x\,dx$

35 $\displaystyle\int \sin^5\frac{x}{4}\,dx$ 36 $\displaystyle\int \tan^4\frac{x}{2}\,dx$ 37 $\displaystyle\int 2\sin^2 ax\cos^2 ax\,dx$

38 $\displaystyle\int \tan^2 x\sec^4 x\,dx$ 39 $\displaystyle\int \cos^4\frac{x}{6}\,dx$

Use either splitting the numerator or algebraic division to find these.

40 $\displaystyle\int \frac{2x - 1}{\sqrt{1 - x^2}}\,dx$ 41 $\displaystyle\int \frac{3x - 4}{\frac{3x^2}{2} - 2x + 3}\,dx$

42 $\displaystyle\int \frac{x^3 + 1}{x - 1}\,dx$ 43 $\displaystyle\int \frac{x^2 + 3}{2x - 1}\,dx$

15.9 Selecting the correct technique 2

The different techniques of integration should now be familiar, but in many situations the technique will not be given. The examples below demonstrate how similar looking questions can require quite different techniques.

Example

Find $\displaystyle\int \frac{3x+2}{3x^2+4x+8}\,dx$.

This is direct reverse beginning with $y = \ln|3x^2+4x+8|$.

So $\dfrac{dy}{dx} = \dfrac{6x+4}{3x^2+4x+8} = \dfrac{2(3x+2)}{3x^2+4x+8}$

$\Rightarrow 2\displaystyle\int \frac{3x+2}{3x^2+4x+8}\,dx = \ln|3x^2+4x+8|$

$\Rightarrow \displaystyle\int \frac{3x+2}{3x^2+4x+8}\,dx = \frac{1}{2}\ln|3x^2+4x+8| + c$

Example

Find $\displaystyle\int \frac{3x+2}{(3x^2+4x+8)^4}\,dx$.

Write the integral as $\displaystyle\int (3x+2)(3x^2+4x+8)^{-4}\,dx$.

This is direct reverse beginning with $y = (3x^2+4x+8)^{-3}$.

So $\dfrac{dy}{dx} = (6x+4)(3x^2+4x+8)^{-4} = 2(3x+2)(3x^2+4x+8)^{-4}$

$\Rightarrow 2\displaystyle\int (3x+2)(3x^2+4x+8)^{-4}\,dx = (3x^2+4x+8)^{-3}$

$\Rightarrow \displaystyle\int (3x+2)(3x^2+4x+8)^{-4}\,dx = \frac{1}{2}(3x^2+4x+8)^{-3} + c$

Example

Find $\displaystyle\int \frac{6x+3}{3x^2+4x+8}\,dx$.

This is a case of splitting the numerator.

Hence the integral becomes $\displaystyle\int \frac{6x+4}{3x^2+4x+8}\,dx - \int \frac{1}{3x^2+4x+8}\,dx$.

The first integral is direct reverse of $y = \ln|3x^2+4x+8|$.

So $\dfrac{dy}{dx} = \dfrac{6x+4}{3x^2+4x+8}$

$\Rightarrow \displaystyle\int \frac{6x+4}{3x^2+4x+8}\,dx = \ln|3x^2+4x+8| + k_1$

The second integral requires completion of the square.

$$\int \frac{1}{3x^2+4x+8}\,dx = \int \frac{1}{3\left(x^2+\frac{4}{3}x+\frac{8}{3}\right)}\,dx$$

$$= \int \frac{1}{3\left[\left(x+\frac{2}{3}\right)^2+\frac{20}{9}\right]}\,dx = \int \frac{1}{3\left(x+\frac{2}{3}\right)^2+\frac{20}{3}}\,dx$$

$$= \frac{1}{\frac{20}{3}}\int \frac{1}{\frac{9}{20}\left(x+\frac{2}{3}\right)^2+1}\,dx = \frac{3}{20}\int \frac{1}{1+\frac{9}{20}\left(x+\frac{2}{3}\right)^2}\,dx$$

Using direct reverse, this begins with $y = \tan^{-1}\dfrac{3}{\sqrt{20}}\left(x+\dfrac{2}{3}\right)$.

$$\frac{dy}{dx} = \frac{1}{1+\frac{9}{20}\left(x+\frac{2}{3}\right)^2} \times \frac{3}{\sqrt{20}}$$

Therefore $\displaystyle\frac{3}{\sqrt{20}}\int \frac{1}{1+\frac{9}{20}\left(x+\frac{2}{3}\right)^2}\,dx = \tan^{-1}\left(x+\frac{2}{3}\right)$

$$\Rightarrow \frac{3}{20}\int \frac{1}{1+\frac{9}{20}\left(x+\frac{2}{3}\right)^2}\,dx = \frac{1}{\sqrt{20}}\tan^{-1}\left(x+\frac{2}{3}\right) + k_2$$

Hence

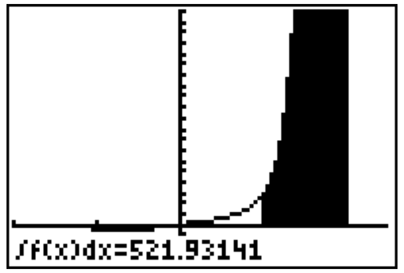
$$\int \frac{6x+3}{3x^2+4x+8}\,dx = \int \frac{6x+4}{3x^2+4x+8}\,dx - \int \frac{1}{3x^2+4x+8}\,dx$$

$$= \ln|3x^2+4x+8| - \frac{1}{\sqrt{20}}\tan^{-1}\left(x+\frac{2}{3}\right) + c$$

Even though all the techniques for this syllabus have been met, there are still a lot of functions that cannot be integrated. There are two reasons for this. First, there are other techniques which have not been covered and second there are some fairly simple looking functions which cannot be integrated by any direct method. An example of this is $\displaystyle\int e^{x^2}\,dx$. However, questions like these in the form of definite integrals can be asked as it is expected that these would be done on a calculator. If a definite integral is asked for on a calculator paper, then it should be done on a calculator unless there is a good reason (for example being asked for an exact answer).

Example

Work out $\int_1^2 x\,e^{x^3}\,dx$.



This cannot be done by any direct method, so the only choice is to use a calculator which will give the answer of 522.

Exercise 9

Use a calculator where appropriate to find these.

1 $\int (x-3)^3\,dx$ **2** $\int \sqrt{3x-5}\,dx$ **3** $\int e^{4x-5}\,dx$

4 $\int (2x^{\frac{2}{3}} - 4x^{\frac{1}{4}})^2\,dx$ **5** $\int \operatorname{cosec} 4x \cot 4x\,dx$ **6** $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^6 x \sin 2x\,dx$

7 $\int x^2 e^{-2x}\,dx$ **8** $\int_0^1 \frac{x^3}{(1+e^x)^{\frac{1}{3}}}\,dx$ **9** $\int \frac{2x-3}{x^2+1}\,dx$

10 $\int \frac{1}{\sqrt{25-4x^2}}\,dx$ **11** $\int \frac{e^x}{(2e^x+1)^3}\,dx$ **12** $\int_0^p \frac{\sin x}{a + b\cos x}\,dx$

13 $\int \frac{x}{(2+5x)^3}\,dx$ using the substitution $u = 2 + 5x$

14 $\int \frac{3}{x^2 - 6x + 25}\,dx$ **15** $\int \cos^4 x\,dx$ **16** $\int_2^5 \frac{3}{5 - 7x^2}\,dx$

17 $\int_0^2 e^{2x^2}\,dx$ **18** $\int \frac{3x^2}{2}\sin 2x\,dx$ **19** $\int \frac{1}{\sqrt{-x^2 - 4x + 29}}\,dx$

20 $\int \log_4 x\,dx$ **21** $\int \frac{1}{\sqrt{x(2-x)}}\,dx$ **22** $\int x^4 \ln 2x\,dx$

23 $\int_1^2 \frac{3x^4}{x^3+3}\,dx$ **24** $\int_{0.1}^{0.5} \frac{\sqrt{3-5x}}{x}\,dx$ **25** $\int e^{-3x}\cos x\,dx$

26 $\int \frac{x^2+7}{5\sqrt{x}}\,dx$ **27** $\int \frac{\tan^4 x}{\cos^4 x}\,dx$ **28** $\int \sqrt{\frac{1-2x}{1+2x}}\,dx$

29 $\int \frac{1}{3 - 2\cos x}\,dx$ using the substitution $t = \tan\frac{x}{2}$

30 $\int_{-1}^{0} \sqrt{4 - 3e^x}\,dx$ **31** $\int_0^a \frac{2x}{(x+1)^4}\,dx$ using the substitution $u = x + 1$

32 $\int \cos^{-1} 2x\,dx$ **33** $\int_0^\infty x\,e^{-x}\,dx$

34 $\int_0^a x^2\sqrt{a^2 - x^2}\,dx$ using the substitution $x = a\sin\theta$

35 $\int_{-1}^{0} \frac{3x^7}{2 - 13x}\,dx$ **36** $\int_1^4 \sin^{-1}\frac{1}{x}\,dx$ **37** $\int \frac{\sin x \cos x}{\cos^2 x - \sin^2 x}\,dx$

38 $\int_0^{\frac{\pi}{3}} \cos 6x \cos 3x\,dx$

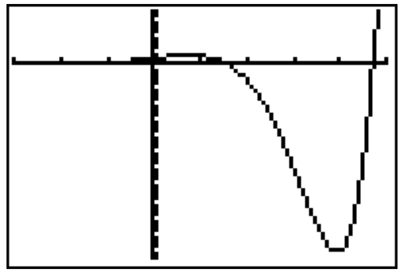
15.10 Finding the area under a curve

We will now look at finding areas under curves by using these techniques.

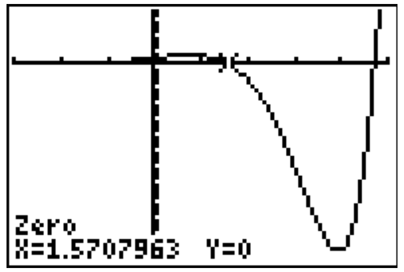
Example

Consider the curve $y = e^x \cos x$. Using a calculator, find the area bounded by the curve, the x-axis, the y-axis and the line $x = a$ where $2 \le a \le 4$.

Drawing the curve on a calculator gives:



To do this question the first point of intersection of the curve with the x-axis needs to be found. Again this is done on a calculator.



Hence the area is given by:

$$A = \left|\int_0^{1.57} e^x \cos x\,dx\right| + \left|\int_{1.57}^{a} e^x \cos x\,dx\right|$$

To find $\int e^x \cos x\,dx$ integration by parts is used.

Letting $\int e^x \cos x\,dx = I$

and using the formula $\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$,

gives $v = e^x$ and $\frac{du}{dx} = \cos x$

$\Rightarrow \frac{dv}{dx} = e^x$ and $u = \int \cos x\,dx = \sin x$

Hence $I = \int e^x \cos x\,dx = e^x \sin x - \int \sin x \times e^x dx = e^x \sin x - \int e^x \sin x dx$

Again using the formula $\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$,

and letting $v = e^x$ and $\frac{du}{dx} = \sin x$

$\Rightarrow \frac{dv}{dx} = e^x$ and $u = \int \sin x\,dx = -\cos x$

Hence $\int e^x \sin x\,dx = -e^x \cos x - \int -\cos x \times e^x\,dx$

$\qquad = -e^x \cos x + \int e^x \cos x\,dx$

Putting it all together $\Rightarrow I = \int e^x \cos x\,dx = e^x \sin x + e^x \cos x - I$

$\Rightarrow I = \frac{1}{2}(e^x \sin x + e^x \cos x) + c$

$$\Rightarrow A = \left|\left[\frac{1}{2}(e^x \sin x + e^x \cos x)\right]_0^{1.57}\right| + \left|\left[\frac{1}{2}(e^x \sin x + e^x \cos x)\right]_{1.57}^{a}\right|$$

$$\Rightarrow A = \left|\left[\frac{1}{2}(e^{1.57} \sin 1.57 + e^{1.57} \cos 1.57) - \frac{1}{2}(e^0 \sin 0 + e^0 \cos 0)\right]\right| + \left|\left[\frac{1}{2}(e^a \sin a + e^a \cos a) - \frac{1}{2}(e^{1.57} \sin 1.57 + e^{1.57} \cos 1.57)\right]\right|$$

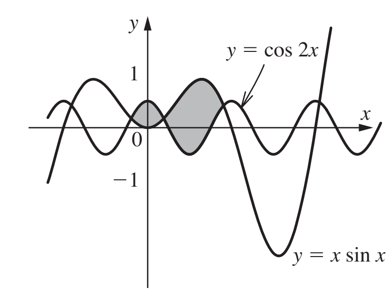
$$\Rightarrow A = \left|2.41 - \frac{1}{2}\right| + \left|\frac{1}{2}(e^a \sin a + e^a \cos a) - 2.41\right|$$

Exercise 10

Use a calculator where appropriate.

1 Calculate the area bounded by the lines $y = 0$, $x = 1$ and the curve $y = \frac{x^2}{x^2 + 1}$.

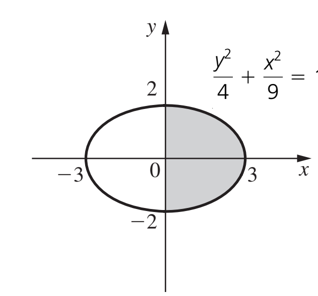
2 Find the area between the curves $y = \cos 2x$ and $y = x \sin x$ shaded in the diagram below.



3 Find the area bounded by the curve $y = xe^{-2x}$, the x-axis and the lines $x = 0$ and $x = 1$.

4 Sketch the curve $y^2 = p^2(p - 2x)$, $p \in \mathbb{R}^+$ and show that the area bounded by the curve and the y-axis is $\frac{2}{3}p^{\frac{5}{2}}$.

5 Find the value of the shaded area in the diagram below which shows the curve.



6 If $y = 3x\sqrt{4 - x^2} + 4 \sin^{-1}\frac{x}{2}$, find $\frac{dy}{dx}$.

Hence show that $\int_0^p \frac{1}{2}\sqrt{4 - x^2}\,dx = 6p\sqrt{4 - p^2} + 8 \sin^{-1}\left(\frac{p}{2}\right)$, $0 \le p \le 2$.

Draw a diagram showing the area this integral represents.

7 Consider the circle $y^2 + x^2 = a^2$ and the line $x = \frac{3a}{4}$. This line splits the circle into two segments. Using integration, find the area of the smaller segment.

8 Show that the exact ratio of $\int_0^{\pi} e^{-x}\cos x\,dx$ to $\int_{2\pi}^{3\pi} e^{-x}\cos x\,dx$ is $-e^{-\pi}$.

9 Show that $\int_2^3 \log_{10} x\,dx$ is $\frac{1}{\ln 10}(3\ln 3 - 2\ln 2 - 1)$.

10 The curve $y = (2x-3)e^{2x}$ crosses the x-axis at P and the y-axis at Q. Find the area bounded by OP, OQ and the curve PQ in terms of e given that O is the origin.

11 Using the substitution $u^2 = 2x - 1$, find the area bounded by the curve $y = x\sqrt{2x-1}$, the x-axis and the lines $x = 1$ and $x = a$, $a > 1$.

12 Find the area bounded by the curve $y = \frac{\sin x}{\sqrt{1-\cos x}}$ and the x-axis.

13 Find the area bounded by the curve $y = x^2 \sin x$ and the x-axis.

14 Find the area between the curves $y = x\sin x$ and $y = e^{3x}$.

Review exercise

1 Evaluate $\int_0^1 (e - ke^{kx})\,dx$.

2 Using the substitution $u = \frac{1}{2}x + 1$, or otherwise, find the integral

$\int x\sqrt{\frac{1}{2}x + 1}\,dx$.

[IB May 99 P1 Q14]

3 Evaluate $\int_4^k \frac{x}{\sqrt{5}}\sqrt{x^2-4}\,dx$.

4 Find $\int \arctan x\,dx$. [IB May 98 P1 Q17]

5 Find the area bounded by the curve $y = \frac{1}{x^2-2x-15}$, the lines $x = -2$, $x = 2$ and the x-axis.

6 Find the real numbers a and b such that $21 + 4x - x^2 = a - (x-b)^2$ for all values of x. Hence or otherwise find $\int \frac{dx}{\sqrt{21+4x-x^2}}$. [IB Nov 88 P1 Q15]

7 The area bounded by the curve $y = \frac{1}{1+4x^2}$, the x-axis and the lines $x = a$ and $x = a + 1$ is 0.1. Find the value of a given that $a > 0$.

8 Find the indefinite integral $\int x^2 e^{-2x}\,dx$. [IB May 97 P1 Q13]

9 **a** Find the equation of the tangent to the curve $y = \frac{1+\ln x}{x}$ which passes through the origin.

b Find the area bounded by the curve, the tangent and the x-axis.

10 Find $\int \frac{dx}{x^2+6x+13}$. [IB Nov 96 P1 Q18]

11 Find $\int \frac{a\cos x}{3 - b\sin x}\,dx$.

12 Let $f: x \mapsto \frac{\sin x}{x}$, $\pi \le x \le 3\pi$. Find the area enclosed by the graph of f and the x-axis. [IB May 01 P1 Q18]

13 For the curve $y = \frac{1}{1+x^2}$:

a find the coordinates of any maximum of minimum points

b find the equations of any asymptotes

c sketch the curve

d find the area bounded by the curve and the line $y = \frac{1}{2}$.

14 Calculate the area bounded by the graph of $y = x\sin(x)^2$ and the x-axis, between $x = 0$ and the smallest positive x-intercept. [IB Nov 00 P1 Q5]

15 Let $f(x) = x\cos 3x$.

a Use integration by parts to show that $\int f(x)\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c$.

b Use your answer to part **a** to calculate the exact area enclosed by $f(x)$ and the x-axis in each of the following cases. Give your answers in terms of π.

i $\frac{\pi}{6} \le x \le \frac{3\pi}{6}$

ii $\frac{3\pi}{6} \le x \le \frac{5\pi}{6}$

iii $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$

c Given that the above areas are the first three terms of an arithmetic sequence, find an expression for the total area enclosed by $f(x)$ and the x-axis for $\frac{\pi}{6} \le x \le \frac{(2n+1)\pi}{6}$, where $n \in \mathbb{Z}^+$. Give your answers in terms of n and π.

[IB May 01 P2 Q1]