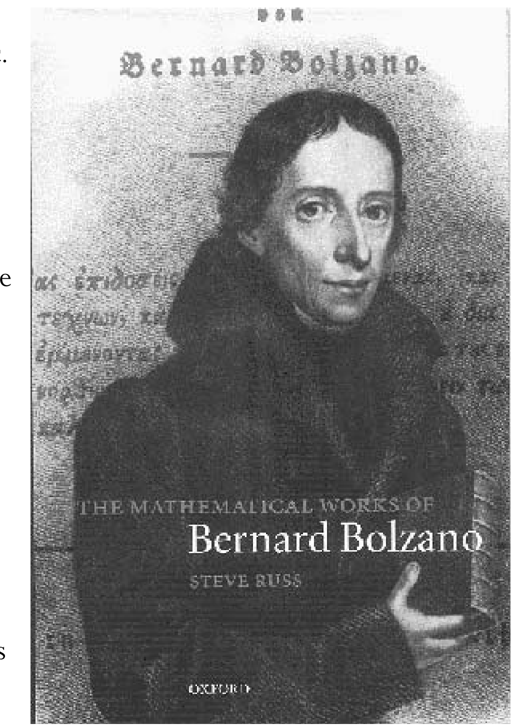
12 Vector Techniques



Bernard Bolzano

Bernard Bolzano was born in 1781 in Prague in what is now the Czech Republic. During Bolzano's early life there were two major influences. The first was his father, who was active in caring for others and the second were the monks who taught him, who were required to take a vow which committed them to take special care of young people. In the year 1799–1800 Bolzano undertook mathematical research with Frantisek Josef Gerstner and contemplated his future. The result of this was that in the autumn of 1800, he went to Charles University to study theology. During this time he also continued to work on mathematics and prepared a doctoral thesis on geometry which led to him publishing a work on the foundations of elementary geometry, *Betrachtungen über einige Gegenstände der Elementargoemetrie* in 1804. In this book Bolzano considers points, lines and planes as undefined elements, and defines operations on them. These are key ideas in the concept of linear space, which then led to the concept of vectors.

Following this, Bolzano entered two competitions for chairs at the Charles University in Prague. One was for the chair of mathematics and the other for the new chair in the philosophy of religion. Bolzano was placed first in both competitions, but the university gave him the chair in the philosophy of religion. In many ways this was the wrong decision, given the way he was brought up with a belief in social justice and pacifism and the fact he was a free thinker. His appointment was viewed with suspicion by the Austrian rulers in Vienna. He criticised the discrimination of the Czech-speaking Bohemians by the German-speaking Bohemians, against their Czech follow citizens and the anti-Semitism displayed by both the German and Czech Bohemians. It came as no surprise that Bolzano was suspended from his position in December 1819 after pressure from the Austrian government. He was also suspended from his professorship, put under house arrest, had his mail censored, and was not allowed to publish. He was then tried by the Church, and was required to recant his supposed heresies. He refused to do so and resigned his chair at the university. From 1823 he continued to study, until in the winter of 1848 he contracted a cold which, given the poor condition of his lungs, led to his death.

12.1 Introduction to vectors

Physical quantities can be classified into two different kinds:

(i) scalar quantities, often called **scalars**, which have magnitude, but no associated direction

(ii) vector quantities, often called **vectors**, which have a magnitude and an associated direction.

So travelling 20 m is a scalar quantity and is called distance whereas travelling 20 m due north is a vector quantity and is called displacement.

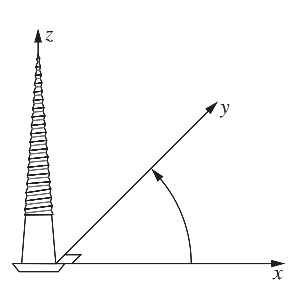
Vector notation

Vectors can be represented in either two or three dimensions, and are described through components. Hence if we want to move from the point (1, 2) on the Cartesian plane to the point (3, 5) we do this by stating we move 2 in the positive x-direction and 3 in the positive y-direction. There are two possible notations for this, column vector notation and unit vector notation.

Column vector notation

In two dimensions a vector can be represented as $\begin{pmatrix} x \\ y \end{pmatrix}$ and in three dimensions as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The conventions for x and y in terms of positive and negative are the same as in the standard two-dimensional Cartesian plane. In three dimensions this is also true, but we need to define what a standard three-dimensional plane looks like. There are three different versions, which are all rotations of each other. In all cases they obey what can be called the "right-hand screw rule". This means that if a screw were placed at the origin and turned with a screwdriver in the right hand from the positive x-axis to the positive y-axis, then it would move in the direction of the positive z-axis. This is shown below.

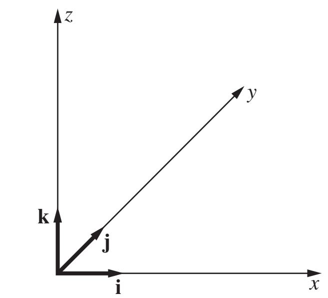


The axes are always drawn like this in this book. Different orientations may be used on IB examination papers.

Unit vector notation

The column vector $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ can be represented as $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ using unit vectors.

Here the unit vectors **i**, **j** and **k** are vectors of magnitude 1 in the directions x, y and z respectively. These are shown in the diagram below.



So the vector $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ means 2 along the x-axis, 3 along the y-axis and -2 along the z-axis.

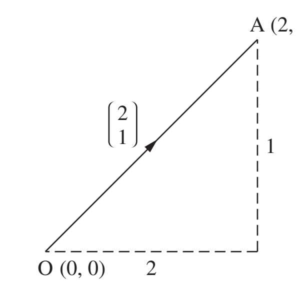
Hence a vector represents a change in position.

There is no advantage to one notation over the other. Both are used in IB examinations and it is probably best to work in the notation given in the question.

Position vectors, free vectors and tied vectors

A vector can be written as a position vector, a free vector or a tied vector.

A **position vector** is one that specifies a particular position in space relative to the origin. For example, in the diagram, the position vector of A is $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

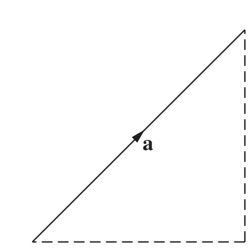


$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ means A is 2 units to the right of O and 1 unit above it.

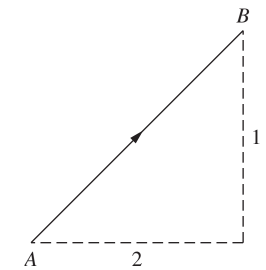
Point A with coordinates (2, 1) has position vector $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

This is true for the position vector of any point.

Now if we talk about a vector $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ then this can be anywhere in space and is therefore a **free vector**.



A vector $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a **tied vector** since it is specified as the vector that goes from A to B.

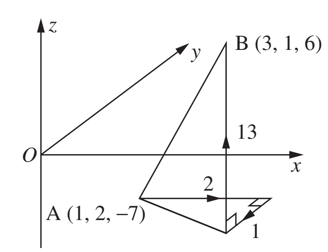


It is obviously possible that $\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and that $\overrightarrow{AB} = \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, but it must be understood that $\overrightarrow{OA}$ and **a** and $\overrightarrow{AB}$ are slightly different concepts.

Whenever vectors are printed in books or in examination papers, free vectors are always written in bold, for example **a**, but in any written work they are written with a bar underneath, $\underline{a}$. Position vectors and tied vectors are always written as the start and end points of the line representing the vector with an arrow above them, for example $\overrightarrow{OA}, \overrightarrow{AB}$.

Forming a tied vector

We now know the vector $\overrightarrow{AB}$ means the vector that takes us from A to B. If we consider A to be the point $(1, 2, -7)$ and B to be the point (3, 1, 6), then to get from A to B we need to move 2 along the *x*-axis, -1 along the *y*-axis and 13 along the *z*-axis. This is shown in the diagram below.



So $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \\ 13 \end{pmatrix}$.

To find $\overrightarrow{BA}$ we subtract the coordinates of B from those of A.

More commonly, we think of $\overrightarrow{AB}$ as being the coordinates of A subtracted from the coordinates of B.

Example

If A has coordinates $(2, -3, 1)$ and B has coordinates $(3, -4, 1)$ find:

a) $\overrightarrow{AB}$

b) $\overrightarrow{BA}$

a) To get from $(2, -3, 1)$ to $(3, -4, 1)$ we go 1 in the *x*-direction, -1 in the *y*-direction and 0 in the *z*-direction. Alternatively, $\overrightarrow{AB} = \begin{pmatrix} 3-2 \\ -4-(-3) \\ 1-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

b) Similarly with $\overrightarrow{BA}$, we go -1 in the *x*-direction, 1 in the *y*-direction and 0 in the *z*-direction. Alternatively, $\overrightarrow{BA} = \begin{pmatrix} 2-3 \\ -3-(-4) \\ 1-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

Notice that in the example

$$\overrightarrow{AB} = -\overrightarrow{BA}$$

This is always true.

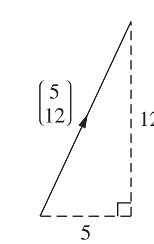
The magnitude of a vector

The magnitude (sometimes called the **modulus**) of a vector is the length of the line representing the vector. To calculate this we use Pythagoras' theorem.

Example

Find the magnitude of the vector $\mathbf{a} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$.

Consider the vector $\mathbf{a} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ in the diagram below.



The magnitude is given by the length of the hypotenuse. $|\mathbf{a}| = \sqrt{5^2 + 12^2} = 13$

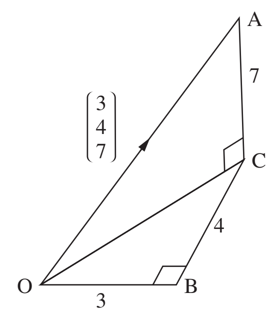
$|\mathbf{a}|$ means the magnitude of **a**.

In three dimensions this becomes a little more complicated.

Example

Find the magnitude of the vector $\begin{pmatrix}3\\4\\7\end{pmatrix}$.

The vector $\begin{pmatrix}3\\4\\7\end{pmatrix}$ is shown in the diagram below. OA is the magnitude of the vector.

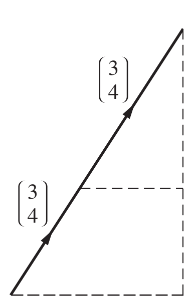


We know by Pythagoras' theorem that $OA = \sqrt{OC^2 + AC^2}$.
Applying Pythagoras' theorem again, $OC^2 = OB^2 + BC^2$.
Hence

$$OA = \sqrt{OB^2 + BC^2 + AC^2}$$
$$\Rightarrow OA = \sqrt{3^2 + 4^2 + 7^2}$$
$$\Rightarrow OA = \sqrt{74}$$

Multiplying a vector by a scalar

When we multiply a vector by a scalar we just multiply each component by the scalar. Hence the vector c hanges in magnitude, but not in direction. For example $2\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix}$ has the same direction as $\begin{pmatrix}3\\4\end{pmatrix}$ but has twice the magnitude. This is shown in the diagram below.



In general, $c\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}3c\\4c\end{pmatrix}$.

There is no symbol for multiplication in this case. This is important as the symbols $\cdot$ and $\times$ have specific meanings in vectors.

Example

If A has coordinates $(2, 3, -1)$ and B has coordinates $(7, 6, -1)$, find these vectors.

a) $\overrightarrow{AB}$

b) $2\overrightarrow{BA}$

c) $p\overrightarrow{AB}$

(a) $\overrightarrow{AB} = \begin{pmatrix}7-2\\6-3\\-1-(-1)\end{pmatrix} = \begin{pmatrix}5\\3\\0\end{pmatrix}$

(b) $\overrightarrow{BA} = \begin{pmatrix}2-7\\3-6\\-1-(-1)\end{pmatrix} = \begin{pmatrix}-5\\-3\\0\end{pmatrix}$

$2\overrightarrow{BA} = 2\begin{pmatrix}-5\\-3\\0\end{pmatrix} = \begin{pmatrix}-10\\-6\\0\end{pmatrix}$

(c) $p\overrightarrow{AB} = p\begin{pmatrix}5\\3\\0\end{pmatrix} = \begin{pmatrix}5p\\3p\\0\end{pmatrix}$

Equal vectors

Vectors are equal if they have the same direction and magnitude.

Example

Find the values of a, b and c for which the vectors $3\begin{pmatrix}a-1\\2b+3\\c\end{pmatrix}$ and $\begin{pmatrix}2a-2\\b+1\\5c-2\end{pmatrix}$ are equal.

If they are equal then

$3(a-1) = 2a-2$
$\Rightarrow 3a - 3 = 2a - 2 \Rightarrow a = 1$
$3(2b+3) = b+1$
$\Rightarrow 6b + 9 = b + 1 \Rightarrow b = -\frac{8}{5}$
$3c = 5c - 2 \Rightarrow c = 1$

Negative vectors

A negative vector has the same magnitude as the positive vector but the opposite direction. Hence if $\mathbf{a} = \begin{pmatrix}2\\-3\end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix}-2\\3\end{pmatrix}$.

Zero vectors

A zero or null vector is a vector with zero magnitude and no directional property. It is denoted by $\begin{pmatrix}0\\0\end{pmatrix}$ or $0\mathbf{i} + 0\mathbf{j}$ in two dimensions. Adding a vector and its negative vector gives the zero vector, i.e. $\mathbf{a} + (-\mathbf{a}) =$ zero vector.

Example

If $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and $2\mathbf{a} + \mathbf{b} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, find $\mathbf{b}$.

$2\mathbf{a} = 4\mathbf{i} + 8\mathbf{j} - 14\mathbf{k}$

$\mathbf{b} = -2\mathbf{a} = -(4\mathbf{i} + 8\mathbf{j} - 14\mathbf{k}) = -4\mathbf{i} - 8\mathbf{j} + 14\mathbf{k}$

In the case of an equation like this the zero vector $0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ could just be written as 0.

If $2\mathbf{a} + \mathbf{b} = 0$ then $\mathbf{b} = -2\mathbf{a}$

Unit vectors

A unit vector is a vector of magnitude one. To find this we divide by the magnitude of the vector. If $\mathbf{n}$ is the vector then the notation for the unit vector is $\hat{\mathbf{n}}$.

Example

Find a unit vector parallel to $\mathbf{m} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$.

The magnitude of $3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ is $\sqrt{3^2 + 5^2 + (-2)^2} = \sqrt{38}$.

Hence the required vector is $\hat{\mathbf{m}} = \frac{1}{\sqrt{38}}(3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$.

Parallel vectors

Since parallel vectors must have the same direction, the vectors must be scalar multiples of each other.

So in two dimensions $\begin{pmatrix}3\\-2\end{pmatrix}$ is parallel to $\begin{pmatrix}15\\-10\end{pmatrix}$ since $\begin{pmatrix}15\\-10\end{pmatrix} = 5\begin{pmatrix}3\\-2\end{pmatrix}$.

In three dimensions $4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ is parallel to $-12\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}$ since $-12\mathbf{i} - 6\mathbf{j} + 15\mathbf{k} = -3(4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$.

Example

Find the value of k for which the vectors $\begin{pmatrix}4\\-2\\8\end{pmatrix}$ and $\begin{pmatrix}12\\-6\\k\end{pmatrix}$ are parallel.

$\begin{pmatrix}4\\-2\\8\end{pmatrix} = 2\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and $\begin{pmatrix}12\\-6\\k\end{pmatrix} = 6\begin{pmatrix}2\\-1\\\frac{k}{6}\end{pmatrix}$

Hence these vectors are parallel when $\frac{k}{6} = 4 \Rightarrow k = 24$.

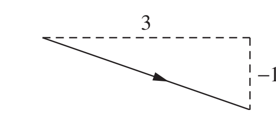
Perpendicular vectors

In the two-dimensional case we use the property that with perpendicular lines the product of the gradients is -1.

Example

Find a vector perpendicular to $\begin{pmatrix}3\\-1\end{pmatrix}$.

From the diagram below we can see that the line representing this vector has a gradient of $-\frac{1}{3}$.



Hence the line representing the perpendicular vector will have a gradient of 3.

Therefore a perpendicular vector is $\begin{pmatrix}1\\3\end{pmatrix}$.

There are an infinite number of perpendicular vectors.

In three dimensions this is more complicated and will be dealt with later in the chapter.

Exercise 1

1 Find the values of a, b and c.

a $\begin{pmatrix}2\\3\\4\end{pmatrix} = 2\begin{pmatrix}a\\b+1\\c-2\end{pmatrix}$ **b** $\begin{pmatrix}1\\b\\-2\end{pmatrix} = \begin{pmatrix}3a\\2b^2\\c+6\end{pmatrix}$ **c** $3\begin{pmatrix}a\\b-1\\4\end{pmatrix} = 4\begin{pmatrix}2-a\\2b+3\\3\end{pmatrix}$

2 If the position vector of P is $\mathbf{i} + \mathbf{j}$ and the position vector of Q is $2\mathbf{i} - 3\mathbf{j}$, find:

a $\overrightarrow{PQ}$ **b** $|\overrightarrow{PQ}|$

3 If the position vector of A is $\begin{pmatrix}2\\3\end{pmatrix}$ and the position vector of B is $\begin{pmatrix}-1\\5\end{pmatrix}$, find:

a $\overrightarrow{AB}$ **b** $|\overrightarrow{AB}|$

4 Write down a vector that is parallel to the line $y = 3x + 5$.

5 Find the magnitude of these vectors.

a $\mathbf{m} = 3\mathbf{i} + 5\mathbf{j}$ **b** $\overrightarrow{OP} = \begin{pmatrix}2\\-7\end{pmatrix}$ **c** $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-3\\-9\end{pmatrix}$

d $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ **e** $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\-1\\2\end{pmatrix}$ **f** $\overrightarrow{OA} = 2\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$

6 State which of the following vectors are parallel to $\begin{pmatrix}1\\-3\\-4\end{pmatrix}$.

a $\begin{pmatrix}4\\-12\\16\end{pmatrix}$ **b** $\frac{1}{3}\mathbf{i} - \mathbf{j} + \frac{4}{3}\mathbf{k}$ **c** $\begin{pmatrix}-5\\-15\\-20\end{pmatrix}$ **d** $0.5p(2\mathbf{i} - 6\mathbf{j} + 8\mathbf{k})$

7 Find the values of c for which the vectors are parallel.

a $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + c\mathbf{j} - 9\mathbf{k}$ **b** $\begin{pmatrix} 14 \\ -35 \\ c \end{pmatrix}$ and $\begin{pmatrix} 18 \\ -45 \\ -9 \end{pmatrix}$

c $\begin{pmatrix} 4t \\ -8t \\ 10t \end{pmatrix}$ and $\begin{pmatrix} ct \\ -12t \\ 15t \end{pmatrix}$

8 A two-dimensional vector has a modulus of 13. It makes an angle of 60° with the x-axis and an angle of 30° with the y-axis. Find an exact value for this vector.

9 Find a unit vector in the direction of $\begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}$.

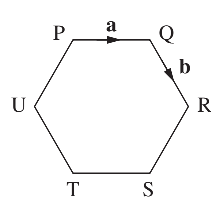
10 A, B, C and D have position vectors given by $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ and $3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$. Determine which of the following pairs of lines are parallel.
a AB and CD **b** BC and CD **c** BC and AD

11 A triangle has its vertices at the points P (1, 2), Q (3, 5) and R(−1, −1). Find the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$, and the modulus of each of these vectors.

12 A parallelogram has coordinates P(0, 1, 4), Q(4, −1, 3), R(x, y, z) and S(−1, 5, 6).
a Find the coordinates of R.
b Find the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{SR}$ and $\overrightarrow{RP}$.
c Find the magnitude of each of the vectors in part **b**.
d Hence write down the unit vectors in the directions of $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{SR}$ and $\overrightarrow{RP}$.

13 If PQRST is a pentagon, show that $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{PT} + \overrightarrow{TS}$.

14 Consider the hexagon shown.



Find the vector represented by $\overrightarrow{SU}$.

15 The vector $\begin{pmatrix} a - b \\ b \end{pmatrix}$ is parallel to the x-axis and the vector $\begin{pmatrix} a + b \\ 2a - b \end{pmatrix}$ is parallel to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find the values of a and b.

16 The vector $(p + 2q - r)\mathbf{i} - (2p + q + r)\mathbf{j} + (3p - 2q - r)\mathbf{k}$ is parallel to $\mathbf{i} + \mathbf{j}$ and the vector $(p + 2r)\mathbf{i} - (p - q + 2r)\mathbf{j} + (p - 2q - 2r)\mathbf{k}$ is parallel to the z-axis. Find the values of p, q and r.

17 If $\overrightarrow{OP} = (3x + 2y)\mathbf{p} + (x - y + 3)\mathbf{q}$, $\overrightarrow{OQ} = (x - y + 2)\mathbf{p} - (2x + y + 1)\mathbf{q}$ and $2\overrightarrow{OP} = 3\overrightarrow{OQ}$, where vectors $\mathbf{p}$ and $\mathbf{q}$ are non-parallel vectors, find the values of x and y.

12.2 A geometric approach to vectors

To add two vectors we just add the x-components, the y-components and the z-components. To subtract two vectors we subtract the x-components, the y-components and the z-components.

Example

If $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}$ find $\mathbf{a} + \mathbf{b}$.

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (2 - 1)\mathbf{i} + (-6 - 6)\mathbf{j} + (12 + 7)\mathbf{k} \\ &= \mathbf{i} - 12\mathbf{j} + 19\mathbf{k}\end{aligned}$$

Example

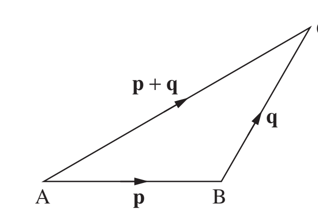
If $\overrightarrow{OA} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -4 \\ -7 \end{pmatrix}$ find $\overrightarrow{OA} - \overrightarrow{OB}$.

$$\overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ -7 \end{pmatrix} = \begin{pmatrix} -3 - 2 \\ 4 - (-4) \\ 2 - (-7) \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 9 \end{pmatrix}$$

We can also look at adding and subtracting vectors geometrically.

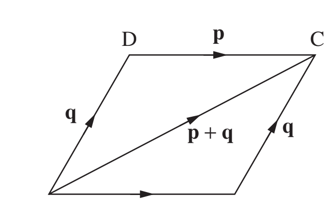
Vector addition

Let the vectors $\mathbf{p}$ and $\mathbf{q}$ be represented by the lines $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively as shown in the diagram.



Then the vector represented by the line $\overrightarrow{AC}$ is defined as the sum of $\mathbf{p}$ and $\mathbf{q}$ and is written as $\mathbf{p} + \mathbf{q}$. This is sometimes called the **triangle law** of vector addition.

Alternatively it can also be represented by a parallelogram. In this case let $\mathbf{p}$ and $\mathbf{q}$ be represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$.



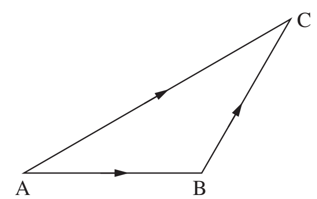
It should be noted that since $\overrightarrow{DC}$ is the same in magnitude and direction as $\overrightarrow{AB}$, the line $\overrightarrow{DC}$ can also represent the vector **p**. Similarly $\overrightarrow{BC}$ can represent the vector **q**.

Comparing this with the triangle, it is clear that the diagonal $\overrightarrow{AC}$ can represent the sum $\overrightarrow{AB} + \overrightarrow{BC}$. This is known as the **parallelogram law** of vector addition.

This also shows that vector addition is commutative.

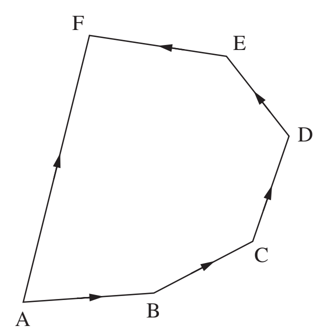
$$\mathbf{q} + \mathbf{p} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{p} + \mathbf{q}$$

Put very simply, vector addition can be thought of as getting from the start point to the end point by any route. Hence in the case of the triangle the route along two sides is the same as the route along the third side because they start and finish at the same point.



Hence $\boxed{\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}}$

This concept can be extended to more than two vectors. To get from A to F we can either go directly from A to F or we can go via B, C, D, and E. Hence $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$.



Example

A quadrilateral has coordinates $A(1, -1, 4)$, $B(3, 2, 5)$, $C(1, 2, 0)$, $D(-1, 2, -1)$. Show that:

a) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

b) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$

a) $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$

$$\overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \overrightarrow{AC}$$

b) $\overrightarrow{CD} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix} = \overrightarrow{AD}$$

Example

If A has coordinates $(0, -1, 2)$ and B has coordinates $(2, -3, 5)$, find:

a) the position vector of the point C, the midpoint of AB

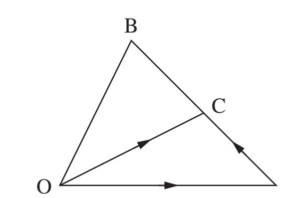
b) the position vector of the point D which divides the line AB in the ratio of 1 : 2.

a)

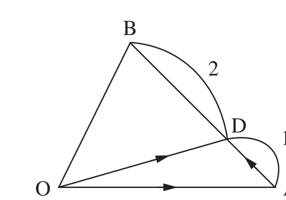
$$\overrightarrow{AB} = \begin{pmatrix} 2-0 \\ -3-(-1) \\ 5-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AC} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ \frac{3}{2} \end{pmatrix}$$

The diagram shows that $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ \frac{7}{2} \end{pmatrix}$.



b) Since the line AB is divided in the ratio of 1 : 2 the point D is $\frac{1}{3}$ the way along the line.



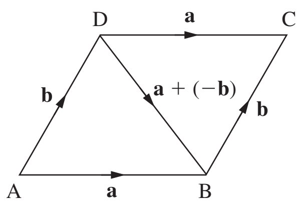
$$\overrightarrow{AD} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{3}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ 1 \end{pmatrix}$$

Therefore the position vector of D is

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-5}{3} \\ 3 \end{pmatrix}.$$

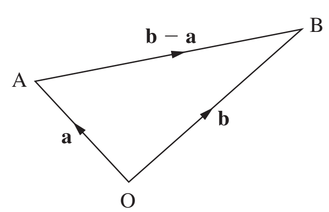
Vector subtraction

We can now use this principle to look at the subtraction of two vectors. We can first consider $\mathbf{a} - \mathbf{b}$ to be the same as $\mathbf{a} + (-\mathbf{b})$.



$\mathbf{a} + (-\mathbf{b})$ is the same as $\overrightarrow{DC} + \overrightarrow{CB}$ and hence the diagonal $\overrightarrow{DB}$ can represent $\mathbf{a} + (-\mathbf{b})$ Thus in terms of the parallelogram one diagonal represents the addition of two vectors and the other the subtraction of two vectors. This explains geometrically why to find $\overrightarrow{AB}$ we subtract the coordinates of A from the coordinates of B.

Alternatively if we consider the triangle below we can see that $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$. Hence $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$.



Example

The diagram shows quadrilateral ABCD, where $\overrightarrow{AB} = 2\mathbf{p}$, $\overrightarrow{DC} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$.

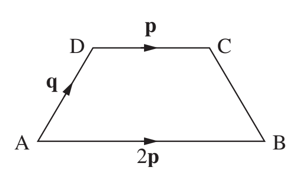
a) What type of quadrilateral is ABCD?

b) Find these in terms of **p** and **q**.

i) $\overrightarrow{BC}$

ii) $\overrightarrow{DB}$

iii) $\overrightarrow{AC}$



a) Since AB and DC are parallel and AD is not parallel to BC, the shape is a trapezium.

b) (i) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC}$

$= -2\mathbf{p} + \mathbf{q} + \mathbf{p} = \mathbf{q} - \mathbf{p}$

(ii) $\overrightarrow{BD} = \overrightarrow{DC} + \overrightarrow{CB}$

$= \mathbf{p} + [-(\mathbf{q} - \mathbf{p})] = 2\mathbf{p} - \mathbf{q}$

(iii) $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

$= \mathbf{q} + \mathbf{p}$

Example

PQRS is a quadrilateral where X and Y are the midpoints of PQ and RS respectively.

Show that $\overrightarrow{PS} + \overrightarrow{SR} = 2\overrightarrow{XY}$.

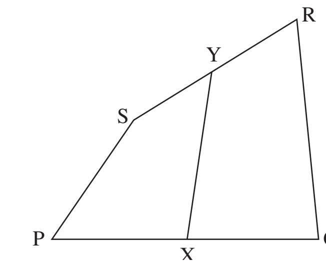
Since $\overrightarrow{PX} = \overrightarrow{XQ}$, $\overrightarrow{XQ} - \overrightarrow{PX} = 0$

$\Rightarrow \overrightarrow{XQ} + \overrightarrow{XP} = 0$

Similarly $\overrightarrow{YR} + \overrightarrow{YS} = 0 \Rightarrow \overrightarrow{RY} + \overrightarrow{SY} = 0$

Now $\overrightarrow{XY} = \overrightarrow{XP} + \overrightarrow{PS} + \overrightarrow{SY}$ and

$\overrightarrow{XY} = \overrightarrow{XQ} + \overrightarrow{QR} + \overrightarrow{RY}$



Hence $2\overrightarrow{XY} = \overrightarrow{XP} + \overrightarrow{PS} + \overrightarrow{SY} + \overrightarrow{XQ} + \overrightarrow{QR} + \overrightarrow{RY}$

$\Rightarrow 2\overrightarrow{XY} = \overrightarrow{PS} + \overrightarrow{QR}$

Exercise 2

1 If $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}$, find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} + \mathbf{b} + \mathbf{c}$ **c** $\mathbf{b} - \mathbf{c}$ **d** $2\mathbf{a} + \mathbf{b} + 4\mathbf{c}$

e $3\mathbf{a} - 3\mathbf{b} - 2\mathbf{c}$ **f** $-2\mathbf{a} + 3\mathbf{b} + 7\mathbf{c}$ **g** $m\mathbf{a} + 20m\mathbf{b} - 3m\mathbf{c}$

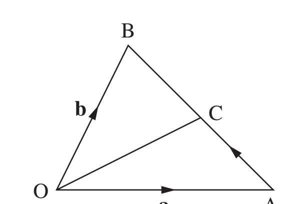
2 If $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = 4\mathbf{i} + 7\mathbf{j}$, find:

a $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$ **b** $|\mathbf{a} - 2\mathbf{b} - 3\mathbf{c}|$

c the angle that $\mathbf{a} - \mathbf{b} + \mathbf{c}$ makes with the x-axis.

3 Vectors **a**, **b**, **c** and **d** are given by $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 7 \\ 5 \\ q \end{pmatrix}$. If $\mathbf{b} - \mathbf{a}$ is parallel to $\mathbf{c} - \mathbf{d}$, find the value of q. Also find the ratio of their moduli.

4 In the triangle shown, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ and C is the midpoint of AB.



Find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{AC}$ **c** $\overrightarrow{CB}$

d Hence, using two different methods, find $\overrightarrow{OC}$.

5 If the position vector of A is $\begin{pmatrix} 4 \\ -16 \end{pmatrix}$ and the position vector of B is $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$, find:

a $\overrightarrow{AB}$ **b** $|\overrightarrow{AB}|$

c the position vector of the midpoint of AB

d the position vector of the point dividing AB in the ratio 2 : 3.

6 If the position vector of P is $\begin{pmatrix} -7 \\ -13 \end{pmatrix}$ and the position vector of Q is $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$, find:

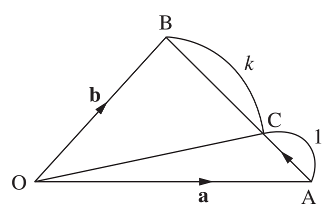
a $\overrightarrow{PQ}$ **b** $|\overrightarrow{PQ}|$

c the position vector of the midpoint of $\overrightarrow{PQ}$

d the position vector of the point dividing $\overrightarrow{PQ}$ in the ratio 1 : 7.

7 In the triangle shown, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point C lies on AB such that AC : CB = 1 : k where k is a constant.
Find:

a $\overrightarrow{AC}$ **b** $\overrightarrow{BC}$ **c** $\overrightarrow{BA}$ **d** $\overrightarrow{OC}$



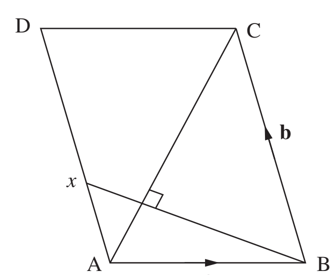
8 ABCD is the parallelogram shown, where $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. $\overrightarrow{AC}$ and $\overrightarrow{BX}$ intersect at 90° and $\overrightarrow{AX} = \frac{1}{3}\overrightarrow{AD}$.

a Find:

i) $\overrightarrow{CD}$ ii) $\overrightarrow{CA}$ iii) $\overrightarrow{BD}$ iv) $\overrightarrow{AX}$ v) $\overrightarrow{XD}$

b If $\mathbf{a} = \begin{pmatrix} 2k \\ 3c \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4k \\ c \end{pmatrix}$, find $\overrightarrow{AC}$ in terms of k and c.

c Hence find, in terms of k and c, a vector parallel to $\overrightarrow{BX}$.

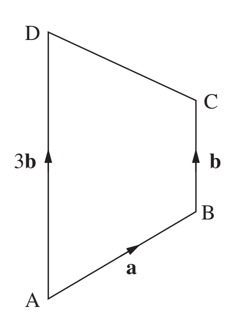


9 The trapezium shown has $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AD} = 3\mathbf{b}$. E and F are points on BC such that BE : EF : FC = m : n : 3.
Find:

a $\overrightarrow{BE}$ **b** $\overrightarrow{EF}$

c $\overrightarrow{CF}$ **d** $\overrightarrow{AF}$

e $\overrightarrow{ED}$

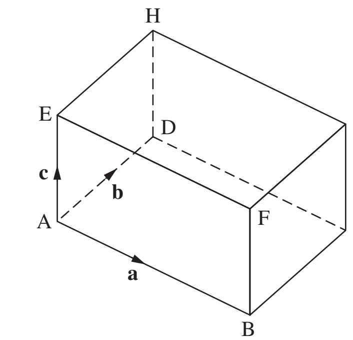


10 The cuboid ABCDEFGH shown has $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AD} = \mathbf{b}$ and $\overrightarrow{AE} = \mathbf{c}$. Find in terms of **a**, **b** and **c**:

a $\overrightarrow{BC}$ **b** $\overrightarrow{FH}$

c $\overrightarrow{AH}$ **d** $\overrightarrow{AG}$

e $\overrightarrow{BH}$



11 If PQR is a triangle and S is the midpoint of PQ, show that $\overrightarrow{QR} + \overrightarrow{PR} = 2\overrightarrow{PS}$.

12 ABCDEFGH is a regular octagon in which $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{CD} = \mathbf{c}$ and $\overrightarrow{DE} = \mathbf{d}$. Find in terms of **a**, **b**, **c** and **d**:

a $\overrightarrow{DG}$ **b** $\overrightarrow{AH}$ **c** $\overrightarrow{FA}$

13 T, U and V are the midpoints of the sides PQ, QR and PS of a triangle. Show that $\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} = \overrightarrow{OT} + \overrightarrow{OU} + \overrightarrow{OV}$, where O is the origin.

14 OABC is a rhombus, where O is the origin, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

a Find $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ and $\overrightarrow{OB}$ in terms of **a** and **c**.

b What is the relationship between $\mathbf{c} + \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$?

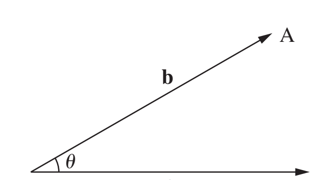
12.3 Multiplication of vectors

When we multiply two vectors there are two possible answers. One answer is a scalar and the other is a vector. Hence one is called the **scalar product** and one is called the **vector product**. We use a "dot" to signify the scalar product and a "cross" to signify the vector product. It is quite common therefore to refer to the "dot product" and "cross product".

The reason why there need to be two cases is best seen through physics. Consider the concept of force multiplied by displacement. In one context this gives the work done, which is a scalar quantity. In another context it gives the moment of a force (the turning effect), which is a vector quantity. Hence in the physical world there are two possibilities, and both need to be accounted for in the mathematical world.

Scalar product

The scalar product or dot product of two vectors **a** and **b** inclined at an angle of θ is written as $\mathbf{a} \cdot \mathbf{b}$ and equals $|\mathbf{a}||\mathbf{b}| \cos \theta$.



"Inclined at an angle of θ" means the angle between the two vectors is θ.

The following results are important.

Parallel vectors

> If two vectors are parallel, then $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 0$ or $|\mathbf{a}||\mathbf{b}|\cos\pi$
>
> So $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$

Perpendicular vectors

> If two vectors are perpendicular, then $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\frac{\pi}{2}$
>
> So $\mathbf{a}\cdot\mathbf{b} = 0$

Commutativity

We know that $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ and $\mathbf{b}\cdot\mathbf{a} = |\mathbf{b}||\mathbf{a}|\cos\theta$.

Since $|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{b}||\mathbf{a}|\cos\theta$ then $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

> The scalar product is commutative.

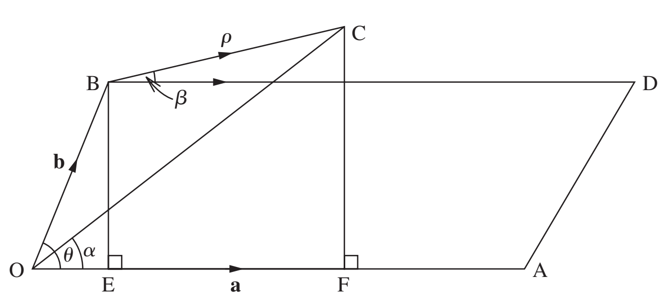
Distributivity

> The scalar product is distributive across addition.

This means $\mathbf{a}\cdot(\mathbf{b}+\boldsymbol{\rho}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\boldsymbol{\rho}$.

Proof

Consider the diagram below, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{BC} = \boldsymbol{\rho}$. $A\hat{O}B = \theta$, $A\hat{O}C = \alpha$, and the angle $\overrightarrow{BC}$ makes with $\overrightarrow{BD}$, which is parallel to $\overrightarrow{OA}$, is β.



Clearly $\overrightarrow{OC} = \mathbf{b} + \boldsymbol{\rho}$

Now $\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\boldsymbol{\rho} = |\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta + |\overrightarrow{OA}||\overrightarrow{BC}|\cos\beta$

$$= |\overrightarrow{OA}|\left(|\overrightarrow{OE}| + |\overrightarrow{EF}|\right)$$

$$= |\overrightarrow{OA}||\overrightarrow{OF}|$$

and $\mathbf{a}\cdot(\mathbf{b}+\boldsymbol{\rho}) = |\overrightarrow{OA}||\overrightarrow{OC}|\cos\alpha = |\overrightarrow{OA}||\overrightarrow{OF}|$

So $\mathbf{a}\cdot(\mathbf{b}+\boldsymbol{\rho}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\boldsymbol{\rho}$, proving the scalar product is distributive over addition.

This result can be extended to any number of vectors.

Scalar product of vectors in component form

Let $\mathbf{p} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{q} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

Then $\mathbf{p}\cdot\mathbf{q} = (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k})\cdot(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$

$$= (a_1a_2\mathbf{i}\cdot\mathbf{i} + b_1b_2\mathbf{j}\cdot\mathbf{j} + c_1c_2\mathbf{k}\cdot\mathbf{k})$$

$$+ (a_1b_2\mathbf{i}\cdot\mathbf{j} + b_1c_2\mathbf{j}\cdot\mathbf{k} + c_1a_2\mathbf{k}\cdot\mathbf{i} + b_1a_2\mathbf{j}\cdot\mathbf{i} + c_1b_2\mathbf{k}\cdot\mathbf{j} + a_1c_2\mathbf{i}\cdot\mathbf{k})$$

We need to look at what happens with various combinations of **i**, **j** and **k**:

$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = (1)(1)\cos 0 = 1$

$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = (1)(1)\cos\frac{\pi}{2} = 0$

Hence

> $\mathbf{p}\cdot\mathbf{q} = a_1a_2 + b_1b_2 + c_1c_2$

So there are two ways of calculating the scalar product. We normally use this form as we rarely know the angle between two vectors.

Example

If $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$, find $\mathbf{a}\cdot\mathbf{b}$.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (3)(2) + (1)(-1) + (-1)(6)\\ &= 6 - 1 - 6\\ &= -1\end{aligned}$$

Example

Given that $\mathbf{x}\cdot\mathbf{p} = \mathbf{q}\cdot\mathbf{x}$ show that **x** is perpendicular to $\mathbf{p} - \mathbf{q}$.

$$\mathbf{x}\cdot\mathbf{p} = \mathbf{q}\cdot\mathbf{x}$$

$$\Rightarrow \mathbf{x}\cdot\mathbf{p} - \mathbf{q}\cdot\mathbf{x} = 0$$

$$\mathbf{x}\cdot\mathbf{p} - \mathbf{x}\cdot\mathbf{q} = 0$$ Since the scalar product is commutative

$$\mathbf{x}\cdot(\mathbf{p} - \mathbf{q}) = 0$$ Using the distributive law

Since the scalar product is zero, **x** and $\mathbf{p} - \mathbf{q}$ are perpendicular.

Example

Show that the triangle ABC with vertices A(1, 2, 3), B(2, −1, 4) and C(3, −3, 2) is not right-angled.

We first write down the vectors representing each side.

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$$

Now $\overrightarrow{AB} \cdot \overrightarrow{BC} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = (1)(1) + (-3)(-2) + (1)(-2) = 1 + 6 - 2 = 5$

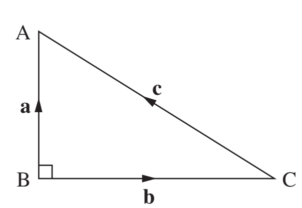
$$\overrightarrow{BC} \cdot \overrightarrow{AC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} = (1)(2) + (-2)(-5) + (-2)(-1) = 2 + 10 + 2 = 14$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ -2 \end{pmatrix} = (1)(2) + (-3)(-5) + (1)(-2) = 2 + 15 - 2 = 15$$

Since none of the scalar products equals zero, none of the sides are at right angles to each other, and hence the triangle is not right-angled.

Example

In the triangle ABC shown, prove that $\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$.



Using the scalar product

$$|\mathbf{c}|^2 = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$

$$\Rightarrow |\mathbf{c}|^2 = |\mathbf{a}|^2 + 0 + 0 + |\mathbf{b}|^2$$

$$\Rightarrow AC^2 = AB^2 + BC^2$$

This is Pythagoras' theorem and hence the relationship is proven.

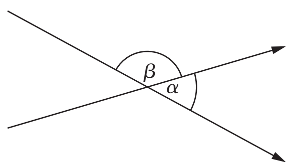
Since AB and BC are at right angles to each other

Angle between two vectors

If we draw two intersecting vectors, there are two possible angles where one is the supplement of the other.

Remember that supplement means "subtract from 180°" or "subtract from π" depending on whether we are working in radians or degrees.

Is the angle between the vectors α or β?



There is a convention for this. The angle between two vectors is the angle between their directions when those directions both converge or both diverge from a point. Hence in this case we require α.

Now we know which angle to find, we can find it using the two formulae for scalar product.

Example

Find the angle θ between $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$.

We know that $\mathbf{a} \cdot \mathbf{b} = (3)(2) + (-2)(4) + (4)(7) = 6 - 8 + 28 = 26$

So $|\mathbf{a}||\mathbf{b}| \cos\theta = 26$

Now $|\mathbf{a}| = \sqrt{3^2 + (-2)^2 + 4^2} = \sqrt{9 + 4 + 16} = \sqrt{29}$

And $|\mathbf{b}| = \sqrt{2^2 + 4^2 + 7^2} = \sqrt{4 + 16 + 49} = \sqrt{69}$

Hence $\cos\theta = \dfrac{26}{\sqrt{29}\sqrt{69}}$

$\Rightarrow \theta = 54.5°$

Example

If the angle between the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ x \end{pmatrix}$ is 60°, find the values of x.

We know that $\mathbf{a} \cdot \mathbf{b} = (3)(2) + (-1)(-1) + (-1)(x) = 6 + 1 - x = 7 - x$

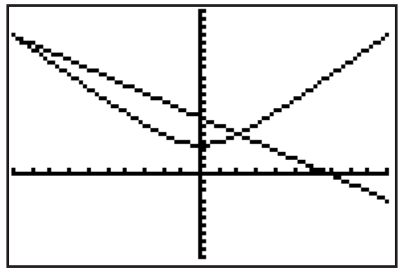
So $7 - x = |\mathbf{a}||\mathbf{b}| \cos 60°$

Now $|\mathbf{a}| = \sqrt{3^2 + (-1)^2 + (-1)^2} = \sqrt{9 + 1 + 1} = \sqrt{11}$

And $|\mathbf{b}| = \sqrt{2^2 + (-1)^2 + x^2} = \sqrt{4 + 1 + x^2} = \sqrt{5 + x^2}$

Hence $7 - x = \sqrt{11}\sqrt{5 + x^2} \cos 60° = \dfrac{\sqrt{11}\sqrt{5 + x^2}}{2}$

This equation cannot be solved by any direct means so we need to use a calculator.



The answer is $x = 2.01$ or -10.0.

Exercise 3

1 Given that $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, find:

a $\mathbf{a} \cdot \mathbf{b}$ **b** $\mathbf{b} \cdot \mathbf{c}$ **c** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ **d** $\mathbf{b} \cdot \mathbf{i}$

e $\mathbf{a} \cdot (\mathbf{c} - \mathbf{b})$ **f** $3\mathbf{a} \cdot \mathbf{c}$ **g** $\mathbf{a} \cdot (2\mathbf{b} + \mathbf{c})$ **h** $\mathbf{b} \cdot (2\mathbf{a} - 3\mathbf{c})$

2 Given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, find:

a $\mathbf{a} \cdot \mathbf{b}$ **b** $\mathbf{c} \cdot \mathbf{b}$ **c** $\mathbf{b} \cdot \mathbf{b}$ **d** $\mathbf{a} \cdot (\mathbf{c} - \mathbf{b})$

e $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{i}$ **f** $\mathbf{b} \cdot (\mathbf{a} + 2\mathbf{b})$ **g** $\mathbf{c} \cdot (\mathbf{a} - 2\mathbf{b})$ **h** $\mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{c}$

3 Calculate the angle between each pair of vectors.

a $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$

b $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

c $\mathbf{a} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$

d $\mathbf{a} = \mathbf{i} + 2\mathbf{k}$, $\mathbf{b} = \mathbf{j} - \mathbf{k}$

e $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

f $\mathbf{a} = \begin{pmatrix} 2t \\ t \\ -3t \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -\frac{1}{t} \\ \frac{2}{t} \\ -\frac{3}{t} \end{pmatrix}$

4 Find $\mathbf{p} \cdot \mathbf{q}$ and the cosine of the angle θ between $\mathbf{p}$ and $\mathbf{q}$ if $\mathbf{p} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + \mathbf{j} - 6\mathbf{k}$.

5 Find which of the following vectors are perpendicular to each other.

$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$,
$\mathbf{d} = -36\mathbf{i} + 27\mathbf{j} - 54\mathbf{k}$, $\mathbf{e} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{f} = 4\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

6 Find the value of λ if the following vectors are perpendicular.

a $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ \lambda \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$

b $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - \lambda\mathbf{k}$

c $\mathbf{a} = \lambda\mathbf{i} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

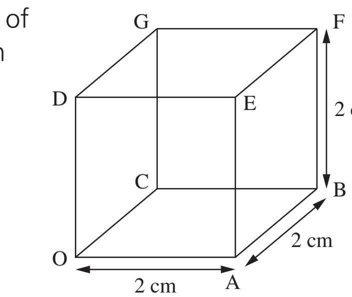
d $\mathbf{a} = \begin{pmatrix} \lambda \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} \lambda \\ \lambda \\ -2 \end{pmatrix}$

7 Show that the triangle ABC is not right-angled, given that A has coordinates $(2, -1, 2)$, B $(3, 3, -1)$ and C $(-2, 1, -4)$.

8 Find a unit vector that is perpendicular to $\overrightarrow{PQ}$ and to $\overrightarrow{PR}$, where $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{QR} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

9 If the angle between the vectors $\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -3 \\ x \\ 4 \end{pmatrix}$ is 80°, find the possible values of x.

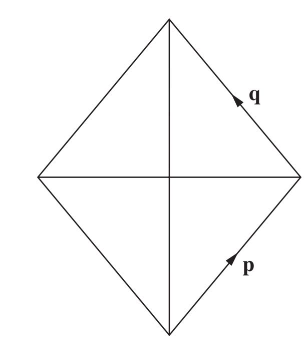
10 Taking O as the origin on a cube OABCDEG of side 2 cm as shown, find the angle between the diagonals OF and AG.



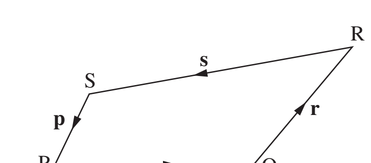
11 A quadrilateral ABCD has coordinates A $(0, 0, 1)$, B $(1, 1, 3)$, C $(3, 0, 6)$ and D $(2, -1, 4)$. Show that the quadrilateral is a parallelogram.

12 A quadrilateral ABCD has coordinates A $(1, 2, -1)$, B $(2, 3, 0)$, C $(3, 5, 3)$ and D $(-2, 3, 2)$. Show that the diagonals of the quadrilateral are perpendicular. Hence state, giving a reason, whether or not the quadrilateral is a rhombus.

13 Using the scalar product, prove that the diagonals of this rhombus are perpendicular to one another.



14 In the quadrilateral PQRS shown, prove that
$\overrightarrow{PR} \cdot \overrightarrow{QS} = \overrightarrow{PQ} \cdot \overrightarrow{RS} + \overrightarrow{QR} \cdot \overrightarrow{PS}$.



15 If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, show that $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$.

16 If A, B, C, D are four points such that $\overrightarrow{BC} + \overrightarrow{DA} = 0$, prove that ABCD is a parallelogram. If $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, state with a reason whether the parallelogram is a rhombus, a rectangle or a square.

17 Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors show that if $|\mathbf{a}| = |\mathbf{b}|$ then $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

18 If **a** and **b** are perpendicular vectors, show that:
$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$

19 Triangle ABC is right-angled at B. Show that $\overrightarrow{AB}\cdot\overrightarrow{AC} = |\overrightarrow{AB}|^2$.

20 Given that **a**, **b** and **c** are non-zero vectors, $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{b}\cdot(\mathbf{a}-\mathbf{c})$, show that $\mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = 0$.

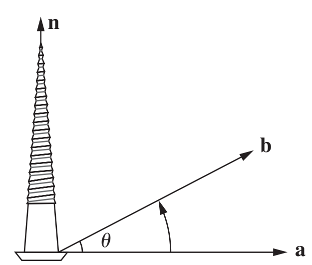
Vector product

The vector product or cross product of two vectors **a** and **b** inclined at an angle θ is written as $\mathbf{a}\times\mathbf{b}$ and equals $|\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$, which is a vector quantity.

Hence it is a vector of magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$ in the direction of **n** where **n** is perpendicular to the plane containing **a** and **b**.

Remember that $\hat{\mathbf{n}}$ is a unit vector.

Now obviously $\hat{\mathbf{n}}$ can have one of two directions. This is decided again by using a "right-hand screw rule" in the sense that the direction of **n** is the direction of a screw turned from **a** to **b** with the right hand. This is shown in the diagram below.



If you are unsure, try this with a screwdriver and a screw!

In other words $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **a** and **b**.

The fact that the vector product of **a** and **b** is perpendicular to both **a** and **b** is very important when it comes to the work that we will do with planes in Chapter 13.

The following results are important.

Parallel vectors

If two vectors are parallel, then $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin 0\hat{\mathbf{n}}$ or $|\mathbf{a}||\mathbf{b}|\sin\pi\hat{\mathbf{n}}$
So $\mathbf{a}\times\mathbf{b} = 0$

Remember that for parallel vectors, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$

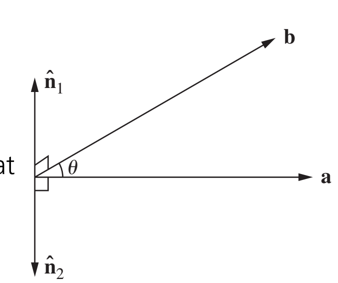
Perpendicular vectors

If two vectors are perpendicular, then $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\frac{\pi}{2}\hat{\mathbf{n}}$
So $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\hat{\mathbf{n}}$

Remember that for perpendicular vectors, $\mathbf{a}\cdot\mathbf{b} = 0$

Commutativity

We know $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}_1$ and that $\mathbf{b}\times\mathbf{a} = |\mathbf{b}||\mathbf{a}|\sin\theta\hat{\mathbf{n}}_2$.



By thinking of the right-hand screw rule we can see that $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ must be in opposite directions.

Hence $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

The vector product is not commutative.

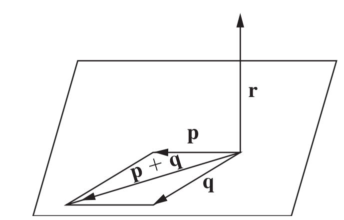
Distributivity

The vector product is distributive across addition.

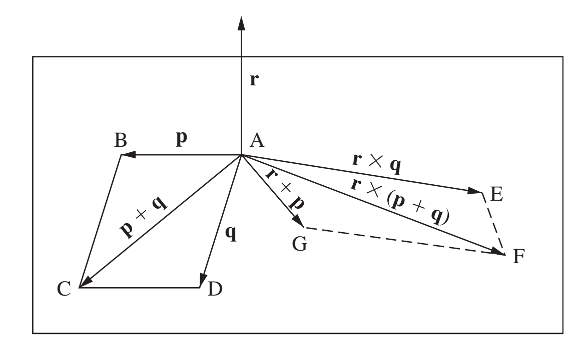
$\mathbf{r}\times(\mathbf{p}+\mathbf{q}) = \mathbf{r}\times\mathbf{p} + \mathbf{r}\times\mathbf{q}$.

Proof

Consider two vectors **p** and **q** with the third vector **r** which is perpendicular to both **p** and **q**. Hence the plane containing **p** and **q** also contains $\mathbf{p}+\mathbf{q}$, and **r** is perpendicular to that plane.



$\mathbf{r}\times\mathbf{p}$, $\mathbf{r}\times\mathbf{q}$ and $\mathbf{r}\times(\mathbf{p}+\mathbf{q})$ must also lie in this plane as all three are vectors that are perpendicular to **r**.



Now $\mathbf{r}\times\mathbf{q} = \overrightarrow{AE} = |\mathbf{r}||\mathbf{q}|\sin 90° = |\mathbf{r}||\mathbf{q}|$

Similarly $\mathbf{r}\times\mathbf{p} = \overrightarrow{AG} = |\mathbf{r}||\mathbf{p}|\sin 90° = |\mathbf{r}||\mathbf{p}|$

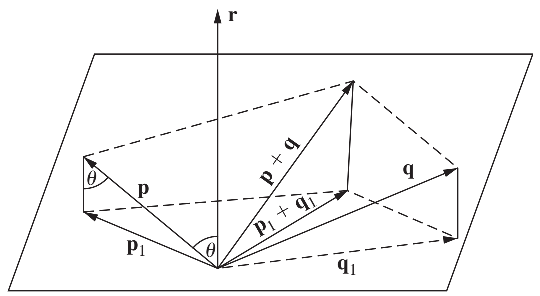
And $\mathbf{r}\times(\mathbf{p}+\mathbf{q}) = \overrightarrow{AF} = |\mathbf{r}||(\mathbf{p}+\mathbf{q})|\sin 90° = |\mathbf{r}||(\mathbf{p}+\mathbf{q})|$

Hence the sides of the quadrilateral AEFG are $|\mathbf{r}|$ times the lengths of the sides in quadrilateral ABCD. Since the angles in both figures are the same ($\mathbf{r}\times\mathbf{p}$ is a 90° rotation of **p**, $\mathbf{r}\times\mathbf{q}$ is a 90° rotation of **q**, and $\mathbf{r}\times(\mathbf{p}+\mathbf{q})$ is a 90° rotation of $\mathbf{p}+\mathbf{q}$), ABCD and AEFG are both parallelograms.

Now we know that $\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF}$

Hence $\mathbf{r}\times(\mathbf{p}+\mathbf{q}) = \mathbf{r}\times\mathbf{p} + \mathbf{r}\times\mathbf{q}$, and we have proved the distributive law when **r** is perpendicular to **p** and **q**.

Now let us consider the case where **r** is not perpendicular to **p** and **q**. In this case we need to form the plane perpendicular to **r** with vectors **p** and **q** inclined at different angles to **r**. $\mathbf{p}_1$, $\mathbf{q}_1$ and $(\mathbf{p}_1+\mathbf{q}_1)$ are the projections of **p**, **q** and $(\mathbf{p}+\mathbf{q})$ on this plane. θ is the angle between **r** and **p**.



Now $\mathbf{r} \times \mathbf{p} = |\mathbf{r}||\mathbf{p}| \sin\theta\hat{\mathbf{n}}$ and $\mathbf{r} \times \mathbf{p}_1 = |\mathbf{r}||\mathbf{p}_1| \sin 90°\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a vector perpendicular to $\mathbf{r}$, $\mathbf{p}$ and $\mathbf{p}_1$.

Also $|\mathbf{p}_1| = |\mathbf{p}| \sin\theta$ and hence $\mathbf{r} \times \mathbf{p}_1 = |\mathbf{r}||\mathbf{p}| \sin\theta\hat{\mathbf{n}} = \mathbf{r} \times \mathbf{p}$.

Using an identical method, $\mathbf{r} \times \mathbf{q}_1 = \mathbf{r} \times \mathbf{q}$ and $\mathbf{r} \times (\mathbf{p}_1 + \mathbf{q}_1) = \mathbf{r} \times (\mathbf{p} + \mathbf{q})$.

Since $\mathbf{r}$ is perpendicular to $\mathbf{p}_1$, $\mathbf{q}_1$ and $\mathbf{p}_1 + \mathbf{q}_1$ we can use the distributive law that we proved earlier, that is, $\mathbf{r} \times (\mathbf{p}_1 + \mathbf{q}_1) = \mathbf{r} \times \mathbf{p}_1 + \mathbf{r} \times \mathbf{q}_1$.

The angle between $\mathbf{r}$ and $\mathbf{p}_1$ is 90° because this is how we set up the plane in the beginning.

Hence $\mathbf{r} \times (\mathbf{p} + \mathbf{q}) = \mathbf{r} \times \mathbf{p} + \mathbf{r} \times \mathbf{q}$ and we have proved that the distributive law also holds when $\mathbf{r}$ is not perpendicular to $\mathbf{p}$ and $\mathbf{q}$.

Vector product of vectors in component form

Let $\mathbf{p} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{q} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

Then $\mathbf{p} \times \mathbf{q} = (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$

$$\begin{aligned} &= (a_1a_2\mathbf{i} \times \mathbf{i} + b_1b_2\mathbf{j} \times \mathbf{j} + c_1c_2\mathbf{k} \times \mathbf{k}) + (a_1b_2\mathbf{i} \times \mathbf{j} + b_1c_2\mathbf{j} \times \mathbf{k} \\ &\quad + c_1a_2\mathbf{k} \times \mathbf{i} + b_1a_2\mathbf{j} \times \mathbf{i} + c_1b_2\mathbf{k} \times \mathbf{j} + a_1c_2\mathbf{i} \times \mathbf{k}) \end{aligned}$$

We need to look at what happens with various combinations of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 1 \times 1 \times \sin 0° \times$ perpendicular vector $= 0$

For the others the answer will always be

$1 \times 1 \times \sin\frac{\pi}{2} \times$ perpendicular vector $=$ perpendicular vector

From the definition of the unit vectors the perpendicular vector to $\mathbf{i}$ and $\mathbf{j}$ is $\mathbf{k}$, to $\mathbf{j}$ and $\mathbf{k}$ is $\mathbf{i}$, and to $\mathbf{i}$ and $\mathbf{k}$ is $\mathbf{j}$. The only issue is whether it is positive or negative, and this can be determined by the "right-hand screw rule". A list of results is shown below.

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$ $\quad$ $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

$\mathbf{j} \times \mathbf{k} = \mathbf{i}$ $\quad$ $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

$\mathbf{k} \times \mathbf{i} = \mathbf{j}$ $\quad$ $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Hence $\mathbf{p} \times \mathbf{q} = a_1b_2\mathbf{k} + b_1c_2\mathbf{i} + c_1a_2\mathbf{j} - b_1a_2\mathbf{k} - c_1b_2\mathbf{i} - a_1c_2\mathbf{j}$

$$= (b_1c_2 - c_1b_2)\mathbf{i} + (c_1a_2 - a_1c_2)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k}$$

$$= (b_1c_2 - c_1b_2)\mathbf{i} - (a_1c_2 - c_1a_2)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k}$$

This can be written as a determinant:

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

So there are two ways of calculating the vector product. We normally use this determinant form as we rarely know the angle between two vectors.

We can use the vector product to calculate the angle between two vectors, but unless there is a good reason, we would normally use the scalar product. One possible reason would be if we were asked to find the sine of the angle between the vectors.

Example

If $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, find
a) the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
b) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$.

a) $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 2 \\ 2 & -4 & 1 \end{vmatrix}$$

$$= \mathbf{i}[3 - (-8)] - \mathbf{j}[1 - 4] + \mathbf{k}[(-4) - 6]$$

$$= 11\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}$$

Hence the unit vector is

$$\frac{1}{\sqrt{11^2 + 3^2 + (-10)^2}}(11\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) = \frac{1}{\sqrt{230}}(11\mathbf{i} + 3\mathbf{j} - 10\mathbf{k})$$

b) We know that $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\hat{\mathbf{n}}$

$$\Rightarrow 11\mathbf{i} + 3\mathbf{j} - 10\mathbf{k} = \sqrt{1^2 + 3^2 + 2^2}\sqrt{2^2 + (-4)^2 + 1^2} \times \sin\theta \times \frac{1}{\sqrt{230}}(11\mathbf{i} + 3\mathbf{j} - 10\mathbf{k})$$

$$\Rightarrow 1 = \sqrt{14}\sqrt{21} \times \sin\theta \times \frac{1}{\sqrt{230}}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{230}}{\sqrt{14}\sqrt{21}} = \sqrt{\frac{230}{294}} = \sqrt{\frac{115}{147}}$$

Example

A, B and C are the points (2, 5, 6), (3, 8, 9), and (1, 1, 0) respectively. Find the unit vector that is perpendicular to the plane ABC.

The plane ABC must contain the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Hence we need a vector perpendicular to two other vectors. This is the definition of the cross product.

Now $\overrightarrow{AB} = \begin{pmatrix} 3-2 \\ 8-5 \\ 9-6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 1-3 \\ 1-8 \\ 0-9 \end{pmatrix} = \begin{pmatrix} -2 \\ -7 \\ -9 \end{pmatrix}$

Therefore the required vector is

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 3 \\ -2 & -7 & -9 \end{vmatrix}$$

$$= \mathbf{i}[(-27) - (-21)] - \mathbf{j}[(-9) - (-6)] + \mathbf{k}[(-7) - (-6)]$$
$$= -6\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Hence the unit vector is

$$\frac{1}{\sqrt{(-6)^2 + 3^2 + (-1)^2}}(-6\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = \frac{1}{\sqrt{46}}(-6\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

Example

If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$, find $\mathbf{c} \cdot \mathbf{a} \times \mathbf{b}$.

In an example like this it is important to remember that we have to do the vector product first, because if we calculated the scalar product first we would end up trying to find the vector product of a scalar and a vector, which is not possible.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 6 & -3 & 2 \end{vmatrix}$$
$$= \mathbf{i}[6 - 3] - \mathbf{j}[4 - (-6)] + \mathbf{k}[(-6) - 18]$$
$$= \begin{pmatrix} 3 \\ -10 \\ -24 \end{pmatrix}$$

Therefore $\mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -10 \\ -24 \end{pmatrix} = 12 - 30 + 24 = 6$

Example

Show that $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = 2\mathbf{b} \times \mathbf{a}$.

Consider the left-hand side.

$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = (\mathbf{a} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{a}) - (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{b})$

Now $\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = 0$

Hence $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = (\mathbf{b} \times \mathbf{a}) - (\mathbf{a} \times \mathbf{b})$

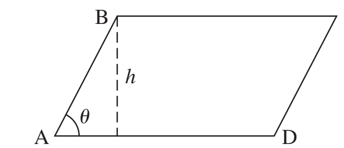
Now we know $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ are the same in magnitude but in opposite directions.

Therefore $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = 2\mathbf{b} \times \mathbf{a}$

Application of vector product

We will see that a very important use of vector products is in the representation of planes, which will be dealt with in Chapter 13. However, there are two other applications that are useful to know.

Area of a parallelogram



The area of a parallelogram $=$ base $\times$ height

$= \text{AD} \times h$

$= (\text{AD})(\text{AB} \sin \theta)$

$= |\overrightarrow{\text{AD}} \times \overrightarrow{\text{AB}}|$

The area of a parallelogram is the magnitude of the vector product of two adjacent sides.

Example

Find the area of the parallelogram ABCD where A has coordinates $(2, 3, -1)$, B $(3, -2, -1)$, C $(-1, 0, -4)$ and D $(3, 0, 1)$.

We first need to find the vectors representing a pair of adjacent sides.

$\overrightarrow{\text{AB}} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix}$ and $\overrightarrow{\text{AD}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

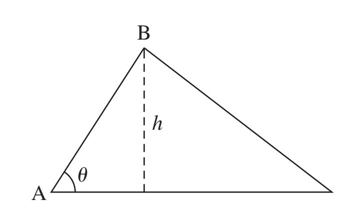
Now $\overrightarrow{\text{AB}} \times \overrightarrow{\text{AD}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -5 & 0 \\ 1 & -3 & 2 \end{vmatrix}$

$$= \mathbf{i}[(-10) - 0] - \mathbf{j}[2 - 0] + \mathbf{k}[(-3) - (-5)]$$
$$= -10\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

Now the area of the parallelogram ABCD is

$|\overrightarrow{\text{AD}} \times \overrightarrow{\text{AB}}| = \sqrt{(-10)^2 + (-2)^2 + 2^2} = \sqrt{108}$ units^2

Area of a triangle



The area of triangle ABC $= \frac{1}{2}$ base $\times$ height

$= \frac{1}{2}(\text{AC})(h)$

$= \frac{1}{2}(\text{AC})(\text{AB} \sin \theta)$

$= \frac{1}{2}|\overrightarrow{\text{AC}} \times \overrightarrow{\text{AB}}|$

The area of a triangle is half the magnitude of the vector product of two sides.

This is consistent with the idea that the area of a triangle is half the area of a parallelogram.

Example

Find the area of the triangle ABC with coordinates $A(1, 3, -1)$, $B(-2, 1, -4)$ and $C(4, 3, -3)$.

We begin by finding $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$$\text{Now } \overrightarrow{AB} = \begin{pmatrix} -2-1 \\ 1-3 \\ -4-(-1) \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 4-(-2) \\ 3-1 \\ -3-(-4) \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{Hence } \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & -3 \\ 6 & 2 & 1 \end{vmatrix}$$

$$= \mathbf{i}[(-2)-(-6)] - \mathbf{j}[(-3)-(-18)] + \mathbf{k}[(-6)-(-12)]$$
$$= 4\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$$

Now the area of triangle ABC is

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}| = \frac{1}{2}\sqrt{4^2 + (-15)^2 + 6^2} = \frac{\sqrt{277}}{2} \text{ units}^2$$

Exercise 4

1 If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, find:

a $\mathbf{a} \times \mathbf{b}$ **b** $\mathbf{a} \times (\mathbf{a} + \mathbf{b})$ **c** $\mathbf{b} \times (\mathbf{a} - \mathbf{b})$

d $\mathbf{a} \times (3\mathbf{a} - 2\mathbf{b})$ **e** $(2\mathbf{a} + \mathbf{b}) \times \mathbf{a}$ **f** $(\mathbf{a} + 2\mathbf{b}) \times (2\mathbf{a} + \mathbf{b})$

g $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$

2 Find the value of $|\mathbf{a} \times \mathbf{b}|$ for the given modulus of **a**, modulus of **b** and angle between the vectors **a** and **b**.

a $|\mathbf{a}| = 3, |\mathbf{b}| = 7, \theta = 60°$ **b** $|\mathbf{a}| = 9, |\mathbf{b}| = \sqrt{13}, \theta = 120°$

c $|\mathbf{a}| = \sqrt{18}, |\mathbf{b}| = 2, \theta = 135°$

3 If OPQ is a triangle with a right angle at P, show that

$|\overrightarrow{OP} \times \overrightarrow{OQ}| = \overrightarrow{OP} \cdot \overrightarrow{PQ}$.

4 If $\mathbf{a} \times \mathbf{b} = 0$, show that $\mathbf{a} = k\mathbf{b}$ where k is a scalar.

5 Given that $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$ show that $\mathbf{a} + \mathbf{b}$ is parallel to **c**.

6 If **a** and **b** are perpendicular, show that $\mathbf{a} \cdot \mathbf{a} \times \mathbf{b} = 0$ irrespective of the values of **a** and **b**.

7 If $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$, find

a the unit vector $\hat{\mathbf{n}}$ perpendicular to both **a** and **b**

b the sine of the angle θ between **a** and **b**.

8 If $\mathbf{a} = \mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{k}$, find

a the unit vector $\hat{\mathbf{n}}$ perpendicular to both **a** and **b**

b the sine of the angle θ between **a** and **b**.

9 P, Q and R are the points $(0, 0, 3)$, $(3, 4, 6)$ and $(0, -1, 0)$ respectively. Find the unit vector that is perpendicular to the plane PQR.

10 Two sides of a triangle are represented by the vectors $(\mathbf{i} + \mathbf{j} - \mathbf{k})$ and $(5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. Find the area of the triangle.

11 Relative to the origin the points A, B and C have position vectors

$\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ respectively. Find the area of the triangle ABC.

12 The triangle ABC has its vertices at the points A (0, 1, 2), B (0, 0, 1) and C (2, 6, 3). Find the area of the triangle ABC.

13 Given that $\overrightarrow{AB} = \mathbf{i} - 4\mathbf{j}$, $\overrightarrow{AC} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\overrightarrow{AP} = 2\overrightarrow{AB}$ and $\overrightarrow{AQ} = 4\overrightarrow{AC}$,

find the area of triangle APQ.

14 A parallelogram OABC has one vertex O at the origin and the vertices A and B at the points (3, 4, 0) and (0, 5, 5) respectively. Find the area of the parallelogram OABC.

15 A parallelogram PQRS has vertices at $P(0, 2, -1)$, $Q(2, -3, -7)$ and $R(-1, 0, -4)$. Find the area of the parallelogram PQRS.

16 A parallelogram PQRS is such that $\overrightarrow{PX} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and

$\overrightarrow{PY} = -3\mathbf{i} - 7\mathbf{j} + \mathbf{k}$, where $\overrightarrow{PQ} = 5\overrightarrow{PX}$ and Y is the midpoint of $\overrightarrow{PS}$.

Find the vectors representing the sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ and hence calculate the area of the parallelogram.

17 If $\mathbf{a} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$, determine whether or not

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

18 If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, show that the vector $\mathbf{c} - \mathbf{b}$ is parallel to **a**.

Review exercise

1 The points P, Q, R, S have position vectors **p**, **q**, **r**, **s** given by

$$\mathbf{p} = \mathbf{i} + 2\mathbf{k}$$
$$\mathbf{q} = -1.2\mathbf{j} + 1.4\mathbf{k}$$
$$\mathbf{r} = -5\mathbf{i} - 6\mathbf{j} + 8\mathbf{k}$$
$$\mathbf{s} = \mathbf{j} - 7\mathbf{k}$$

respectively. The point X lies on PQ produced and is such that $PX = 5PQ$, and the point Y is the midpoint of PR.

a Show that XY is perpendicular to PY.

b Find the area of the triangle PXY.

c Find a vector perpendicular to the plane PQR.

d Find the cosine of the acute angle between PS and RS.

2 Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ p \\ 6 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$

a Find $\mathbf{a} \times \mathbf{b}$

b Find the value of p, given that $\mathbf{a} \times \mathbf{b}$ is parallel to **c**. [IB May 06 P1 Q11]

3 The point A is given by the vector $\begin{pmatrix} 1 - m \\ 2 + m \\ 3 + m \end{pmatrix}$ and the point B by $\begin{pmatrix} 1 - 2m \\ 2 + 2m \\ 3 + 2m \end{pmatrix}$, relative to O. Show that there is no value of m for which $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular.

4 If **a** and **b** are unit vectors and θ is the angle between them, express $|\mathbf{a} - \mathbf{b}|$ in terms of θ. [IB May 93 P1 Q12]

5 A circle has a radius of 5 units with a centre at (3, 2). A point P on the circle has coordinates (x, y). The angle that this radius makes with the horizontal is θ. Give a vector expression for $\overrightarrow{OP}$.

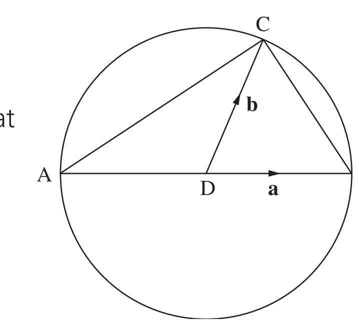
6 Given two non-zero vectors **a** and **b** such that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, find the value of $\mathbf{a} \cdot \mathbf{b}$. [IB Nov 02 P1 Q18]

7 Show that the points A(0, 1, 3), B(5, 3, 2) and C(15, 7, 0) are collinear (that is, they lie on the same line).

8 Find the angle between the vectors $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Give your answer in radians. [IB May 02 P1 Q5]

9 The circle shown has centre D, and the points A, B and C lie on the circumference of the circle. The radius of the circle is 1 unit.

Given that $\overrightarrow{DB} = \mathbf{a}$ and $\overrightarrow{DC} = \mathbf{b}$, show that $\hat{ACB} = 90°$.



10 Let α be the angle between **a** and **b**, where

$\mathbf{a} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, $\mathbf{b} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ and $0 < \theta < \frac{\pi}{4}$. Express α in terms of θ. [IB Nov 00 P1 Q11]

11 The points X and Y have coordinates (1, 2, 3) and $(2, -1, 0)$ respectively. $\overrightarrow{OA} = 2\overrightarrow{OX}$ and $\overrightarrow{OB} = 3\overrightarrow{OY}$. OABC is a parallelogram.

a Find the coordinates of A, B and C.

b Find the area of the parallelogram OABC.

c Find the position vector of the point of intersection of $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

d The point E has position vector **k**. Find the angle between $\overrightarrow{AE}$ and $\overrightarrow{BE}$.

e Find the area of the triangle ABE.

12 Given $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + m\mathbf{k}$, and that the vector $(2\mathbf{u} - 3\mathbf{v})$ has magnitude $\sqrt{265}$, find the value of m. [IB Nov 93 P1 Q8]