0 Presumed Knowledge

0.1 Sets

Number systems

The number systems that it is important to be aware of are as follows:

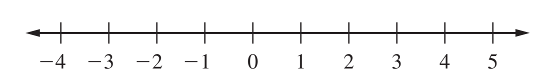
The natural numbers, denoted by $\mathbb{N}$, are $\{0, 1, 2, 3, 4, \ldots\}$.

The integers, denoted by $\mathbb{Z}$, are $\{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

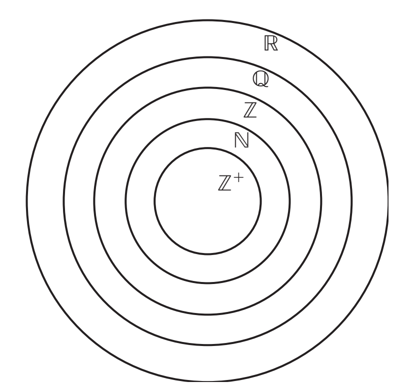
The positive integers, denoted by $\mathbb{Z}^+$, are $\{1, 2, 3, 4, \ldots\}$.

The rational numbers, denoted by $\mathbb{Q}$, are defined as the set of numbers that can be expressed as a fraction of two integers, i.e. $\left\{x : x = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0\right\}$.

The real numbers, denoted by $\mathbb{R}$, can be considered as any number on the number line including irrational numbers such as $\sqrt{2}$, π.



The diagram below shows the relationship between these sets:



Set notation

Within this syllabus, set notation will be used. A set is a collection of objects; for this syllabus sets tend to be collections of numbers.

$\{\}$ These brackets denote a set.
$\varnothing$ denotes the empty set.
$\in$ means "is a member of".
ε denotes the universal set, which means the set of numbers upon which other sets are defined.
$\cap$ denotes the intersection of two sets. Intersection means the members of the two sets that are elements in both sets. This is analogous to "and".

$\cup$ denotes the union of two sets. Union means all the members that are in either set. This is analogous to "or".
$\subset$ or $\subseteq$ means "is a subset of". This is analogous to $<$ and $\leq$ for numbers.

Example

What does $x \in \{3, 4, 7\}$ mean?

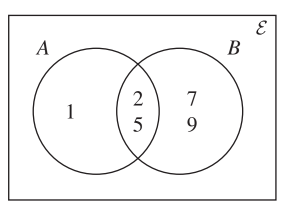
$x \in \{3, 4, 7\}$ means that x is a member of the set containing 3, 4 and 7.

Venn diagrams

We can represent sets using a Venn diagram, where a rectangle represents the universal set, and circles inside the rectangle represent the subsets of the universal set.

Example

Draw $A = \{1, 2, 5\}$ and $B = \{2, 5, 7, 9\}$ on a Venn diagram and write down the intersection and union of A and B.

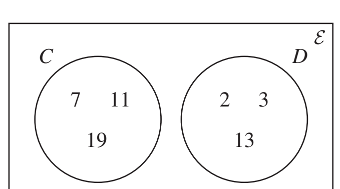


The sets A and B overlap as an intersection exists. The intersection is $A \cap B = \{2, 5\}$.

The union of the sets is $A \cup B = \{1, 2, 5, 7, 9\}$.

Example

Draw $C = \{7, 11, 19\}$ and $D = \{2, 3, 13\}$ on a Venn diagram and write down the intersection and union of C and D.



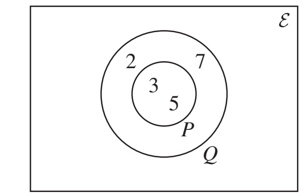
Here the sets do not overlap so $C \cap D = \varnothing$.

$C \cup D = \{2, 3, 7, 11, 13, 19\}$

0 Presumed Knowledge

Example

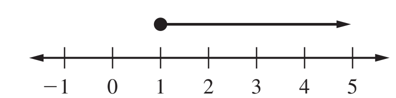
Draw $P = \{3, 5\}$ and $Q = \{2, 3, 5, 7\}$ on a Venn diagram and write down the intersection and union of P and Q. State why P is a subset of Q.



$P \cap Q = \{3, 5\}$ and $P \cup Q = \{2, 3, 5, 7\}$. Here all of set P is contained in set Q and so P is a subset of Q, that is, $P \subset Q$.

It is important to be able to express an inequality as a set.

Example

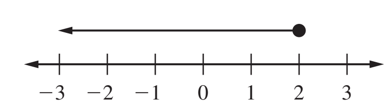


This can be expressed as the set $\{x : x \geq 1, x \in \mathbb{R}\}$.
This means the set of numbers x, where x is greater than or equal to 1, where x is a member of the real numbers.

Exercise 1

1 Express each inequality as a set.

a



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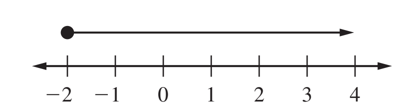
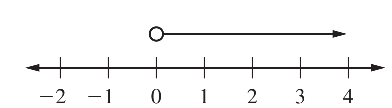
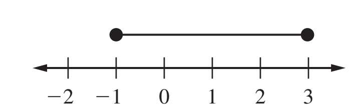


c



d



2 For the sets given, draw a Venn diagram and state the intersection and union of the sets.

a $A = \{1, 2, 3\}, B = \{3, 4, 5\}$ **b** $C = \{0, 3, 7\}, D = \{3, 9, 11, 13\}$

c $F = \{-3, -2, 10\}, G = \{1, 2, 7\}$ **d** $K = \left\{\frac{1}{2}, \frac{3}{4}, 9\right\}, L = \left\{\frac{1}{2}, 9, 11, 17\right\}$

e $P = \{-5, 1, 2\}, Q = \{-5, 1, 2, 7, 11\}$

3 If $B = \{3, 7, 9, 11, 13\}, A \subset B$ and $A \cap B = \{7, 9, 11\}$, state the set A. Draw a Venn diagram of these two sets.

4 If $P = \{-5, 3, 7, 11\}$, $Q = \{-2, 3, 5, 7, 9, 11\}$ and $R = \{-5, 0, 1, 7\}$, draw a Venn diagram of these sets. State $P \cup Q$ and $P \cap Q \cap R$.

5 If $A = \{-1, 3, 5, 6\}$, $B = \{-1, 3, 5\}$ and $C = \{-1, 0, 1, 7, 9\}$, draw a Venn diagram of these sets. State $B \cup C$ and $A \cap B \cap C$.

0.2 The Cartesian plane

Drawing a straight-line graph

Straight-line graphs have a general equation $y = mx + c$ where m denotes the gradient and c denotes the y-intercept. Gradient is a measure of steepness, and a straight line has a constant (unchanging) gradient.

To draw a straight-line graph, choose three x-values (two is the minimum but a third value is wise to check) and find the corresponding y-values. Plot the points and join them using a ruler.

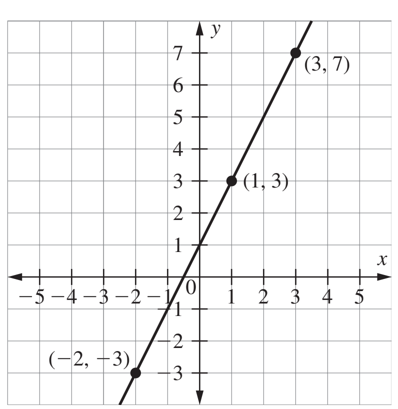
Example

Draw the graph of $y = 2x + 1$.

Choosing $x = -2$, $x = 1$ and $x = 3$:

$y = 2(-2) + 1 = -3$ $\quad$ $y = 2(1) + 1 = 3$ $\quad$ $y = 2(3) + 1 = 7$

So we have three points: $(-2, -3)$, $(1, 3)$, $(3, 7)$

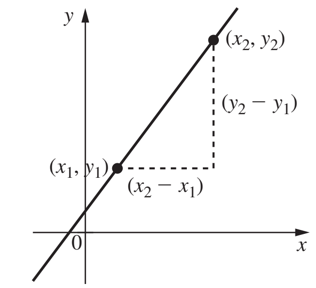


We can see that the y-intercept is (0, 1).

Finding the gradient between two points

The gradient between two points (x_1, y_1) and (x_2, y_2) is defined to be

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

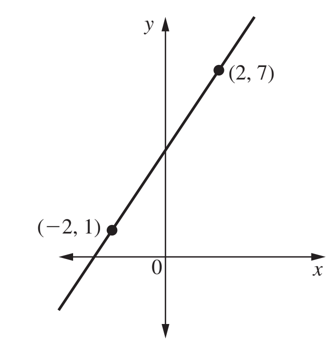


This formula states that the gradient is $\dfrac{\text{vertical shift}}{\text{horizontal shift}}$.

Example

Find the gradient of the line joining $(-2, 1)$ and $(2, 7)$.

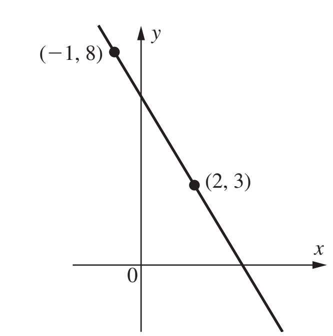
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 1}{2 - (-2)} = \frac{6}{4} = \frac{3}{2}$$



Example

Find the gradient of the line joining $(-1, 8)$ and $(2, 3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 8}{2 - (-1)} = \frac{-5}{3}$$



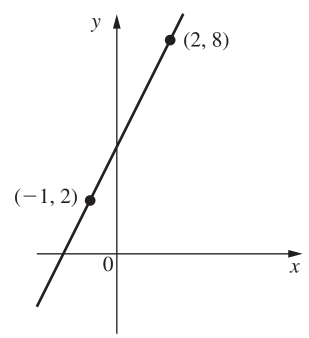
A negative gradient signifies that the line has a downwards direction (from left to right).

Finding the equation of a straight line

To find the equation of a straight line, we need to find the gradient and the y-intercept.

Example

Find the equation of this line.



Here the gradient is $m = \dfrac{8-2}{2-(-1)} = \dfrac{6}{3} = 2.$

So the equation of the line is of the form $y = 2x + c$. We now need to find c. This can be done in two ways.

Method 1

We can substitute a point into the equation to find c.
Substituting (2, 8) in $y = 2x + c$:

$$8 = 2 \times 2 + c$$
$$\Rightarrow 8 = 4 + c$$
$$\Rightarrow c = 4$$

So $y = 2x + 4$

Method 2

We can use the formula $y - b = m(x - a)$ where (a, b) is a point on the line. This is just a general form of $y = mx + c$ found by substituting (a, b) into the equation:

$$b = ma + c$$
$$\Rightarrow c = b - ma$$

Now substituting this into $y = mx + c$, we have

$$y = mx + b - ma$$

Rearranging gives $y = mx + b - ma$

$$\Rightarrow y - b = mx - ma$$
$$\Rightarrow y - b = m(x - a)$$

So choosing (2, 8), we have

$$y - 8 = 2(x - 2)$$
$$\Rightarrow y - 8 = 2x - 4$$
$$\Rightarrow y = 2x + 4$$

Example

Find the equation of the line joining (1, 9) and (4, 5).

$$m = \frac{5-9}{4-1} = -\frac{4}{3}$$

$$\text{So } y - 9 = -\frac{4}{3}(x - 1)$$
$$\Rightarrow y - 9 = -\frac{4}{3}x + \frac{4}{3}$$
$$\Rightarrow y = -\frac{4}{3}x + \frac{31}{3}$$

Other forms of the straight-line equation

Although it is very useful to have a straight-line equation in the form $y = mx + c$ (because the gradient and y-intercept are immediately apparent), a straight line is sometimes expressed in the form $ax + by + c = 0$. This form has the advantage of eliminating any fractions.

Example

Find the equation of the line joining $(-2, -4)$ and $(1, 3)$ in the form $ax + by + c = 0$, where a, b and c are constants.

$$m = \frac{3+4}{1+2} = \frac{7}{3}$$

So, using $(-2, -4)$,

$$y + 4 = \frac{7}{3}(x + 2)$$

Multiply both sides by 3 to remove the fraction.

$$\Rightarrow 3y + 12 = 7(x + 2)$$
$$\Rightarrow 3y + 12 = 7x + 14$$
$$\Rightarrow 3y - 7x - 2 = 0$$

This can then be rearranged to $y = \dfrac{7}{3}x + \dfrac{2}{3}$ if the $y = mx + c$ form is required.

A calculator can be used to sketch straight-line graphs but they must be in the form $y = f(x)$.

Parallel and perpendicular lines

Parallel lines have the same gradient. We know that in the equation $y = mx + c$, the gradient determines the direction and the y-intercept determines the position of the line on the Cartesian grid. So two lines with the same direction have the same gradient (and are parallel).

Example

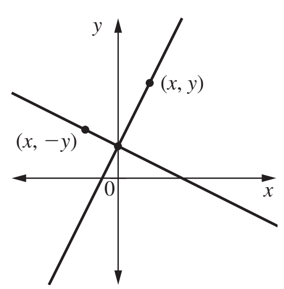
Find the line parallel to $y = 3x - 2$ that passes through (4, 1).

Since the line is parallel, $m = 3$.
So the line that we want is given by $y - 1 = 3(x - 4)$

$$\Rightarrow y - 1 = 3x - 12$$
$$\Rightarrow y = 3x - 11$$

Two lines that are perpendicular have the property that $m_1 m_2 = -1$.

To find the reason for this, consider the effect of rotating a point 90° anticlockwise about the origin, i.e. $(x, y) \rightarrow (-y, x)$.



Applying this to $y = mx + c$:

$$x = m(-y) + c$$
$$\Rightarrow my = -x + c$$
$$\Rightarrow y = -\frac{1}{m}x + \frac{c}{m}$$

So the gradients of perpendicular lines are m and $-\frac{1}{m}$. Multiplying these gives $m\left(-\frac{1}{m}\right) = -1$.

Example

Find the equation of the line that is perpendicular to $y = 3x + 2$ and passes through (2, 3).

Since the line is perpendicular to $y = 3x + 2$

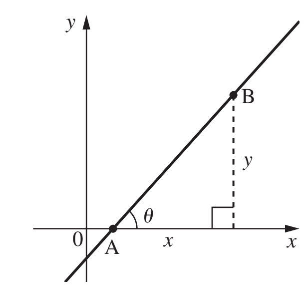
$$m \times 3 = -1$$
$$\Rightarrow m = -\frac{1}{3}$$

So the line has equation

$$y - 3 = -\frac{1}{3}(x - 2)$$
$$\Rightarrow y - 3 = -\frac{1}{3}x + \frac{2}{3}$$
$$\Rightarrow y = -\frac{1}{3}x + \frac{11}{3}$$

Gradient and trigonometry

If the scales on the x- and y-axes are the same, then the line provides a visual representation of the gradient, in particular the angle with the x-axis. Considering this from a trigonometrical perspective:



Looking at the line AB in the diagram above, we know that $m = \frac{y}{x}$. We also know from right-angled trigonometry that $\tan\theta = \frac{y}{x}$.

Hence $m = \tan\theta$.

Example

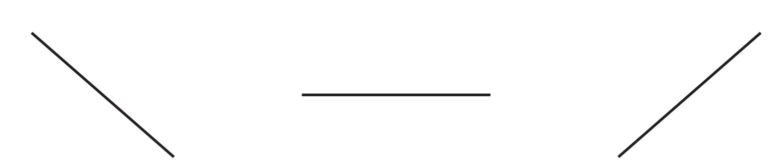
Find the angle that the line $y = 2x + 3$ makes with the positive direction of the x-axis.

Since $m = 2$

$$\tan\theta = 2$$
$$\Rightarrow \theta = \tan^{-1}(2)$$
$$\Rightarrow \theta = 63.4°$$

Horizontal lines

A horizontal line has zero gradient.



Negative gradient $\quad$ $m = 0$ $\quad$ Positive gradient

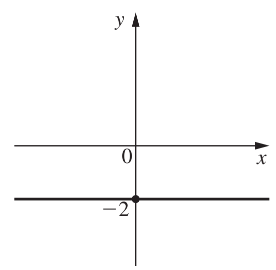
This means that horizontal lines have equations of the form $y = 0x + c$

$$\Rightarrow y = c$$

Thus the y-coordinate of every point on the line is the same regardless of the x-coordinate.

Example

What is the equation of this line?

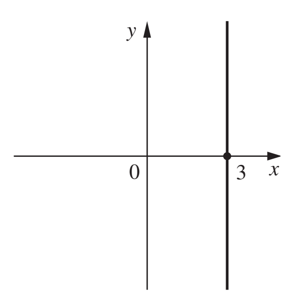


$y = -2$

Considering a horizontal line in this way provides the form for a vertical line, which is $x = k$.

Example

State the equation of this line.



$x = 3$

For a vertical line, the gradient is undefined. This can be seen by taking any two points on the line, e.g. (3, 1) and (3, 4). $m = \frac{4-1}{3-3} = \frac{3}{0}$, which is undefined. We can also consider this from a trigonometrical perspective as tan 90° is undefined.

Midpoint of two points

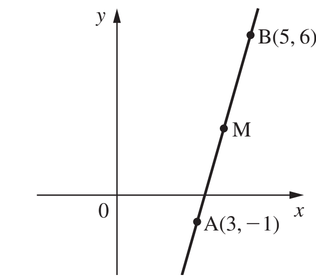
The midpoint of two points is found by averaging the x-coordinates and the y-coordinates. This means that the midpoint M of (x_1, y_1) and (x_2, y_2) is given by

$$\text{Midpoint, M} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example

Find the midpoint of A(3, −1) and B(5, 6).

$$M_{AB} = \left(\frac{3+5}{2}, \frac{-1+6}{2}\right)$$
$$= \left(4, \frac{5}{2}\right)$$



Distance between two points

To find the distance between two coordinate points, we create a right-angled triangle and apply Pythagoras' theorem. This is demonstrated in the example below.

Example

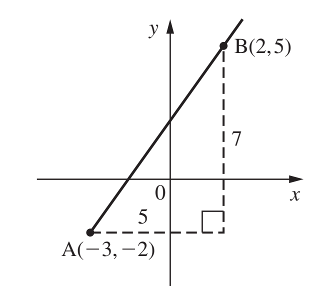
Find the distance from A to B.

So $AB^2 = 5^2 + 7^2$

$\Rightarrow AB^2 = 74$

$\Rightarrow AB = \sqrt{74}$

$\Rightarrow AB = 8.60$



This process is sometimes given as the distance formula

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Exercise 2

1 Find the gradient of the line joining each pair of points.

a A(2, 3) and B(4, 7) **b** C(−1, 2) and D(3, 7)
c E(−3, −2) and F(1, 7) **d** G(−1, 8) and H(2, 1)
e I(3, 8) and J(7, 0) **f** K(−1, 2) and L(3, 2)

2 Draw the graph of these straight lines.

a $y = 2x - 3$ **b** $y = 3x + 1$
c $y = -2x + 5$ **d** $y = 8 - x$
e $2x + 3y - 7 = 0$

3 Find the equation of the line joining each pair of points.

a A(1, 4) and B(4, 11) **b** C(−1, 2) and D(4, 5)
c E(−4, −1) and F(−2, 5) **d** G(−1, 8) and H(1, 4)
e I(−7, 9) and J(−3, 6) **f** K(−4, −2) and L(1, −2)

4 Find the equation of the line joining each pair of points, in the form $ax + by + c = 0$.

a $P(-4, 2)$ and $Q(-1, 3)$ **b** $R(-3, 7)$ and $S(-1, 0)$

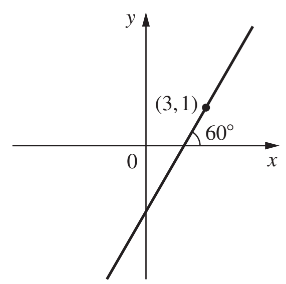
c $T(-5, -7)$ and $U(-1, -2)$

5 Find the angle each straight line makes with the positive direction of the x-axis.

a $y = x + 7$ **b** $y = 3x - 1$

c $y = 9 - 2x$ **d** $4x - 3y + 9 = 0$

6 Find the equation of the line below.



7 Find the equation of the line parallel to $y = 3x - 4$ that passes through (2, 4).

8 Find the equation of the line perpendicular to $y = 4x - 3$ that passes through (1, 3).

9 Find the equation of the line perpendicular to $y = 9 - \frac{1}{2}x$ that passes through (2, 5).

10 Find the midpoint of the line joining each pair of points.

a A(2, 1) and B(6, 7) **b** $C(-1, 2)$ and D(3, 7)

c $E(-5, -2)$ and $F(-1, -7)$ **d** G(1, 2) and H(7, 2)

e $I(-5, 11)$ and $J(1, -2)$

11 Find the distance between the pairs of points in question 10.

12 A is the point $(-5, -3)$ and B is (1, 9).

a Calculate the midpoint of AB, M.

b Find the distance AB.

c Find the equation of the line joining A and B.

d Find the equation of the line that is perpendicular to AB and passes through M.

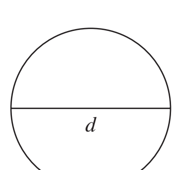
e What angle does this line make with the positive direction of the x-axis?

0.3 Circle properties

Circumference of a circle

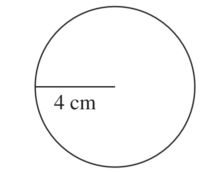
The circumference (perimeter) of a circle and the diameter of the circle are directly proportional. This is the definition of the constant π and allows us to find the circumference of a circle.

$$C = \pi d$$



Example

Find the circumference of this circle.



Since $r = 4$, $d = 8$

So $C = \pi d$

$= \pi \times 8$

$= 25.1$ cm

Area of a circle

The sectors of a circle can be rearranged to form a shape that is nearly a rectangle.

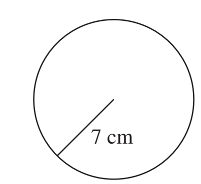
The more sectors there are, the closer the shape is to a rectangle.

So $A = r \times \pi r$

$$A = \pi r^2$$

Example

Find the area of this circle.

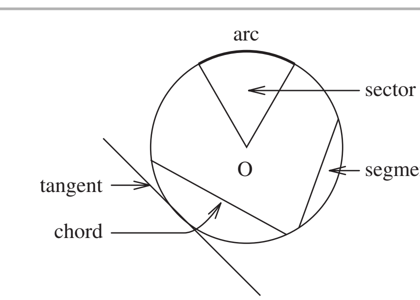


$A = \pi r^2$

$\Rightarrow A = \pi \times 7^2$

$\Rightarrow A = 154 \text{ cm}^2$

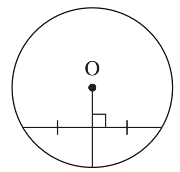
Circle notation



Chord – A line joining two points on the circumference of a circle.
Arc – Part of the circumference.
Sector – Part of the area of a circle (a slice).
Segment – The area between a chord and the circumference.
Tangent – A line that touches the circle at one point.

Chord properties

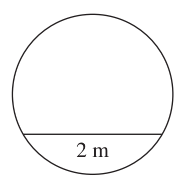
The radius that bisects the chord is perpendicular to the chord.



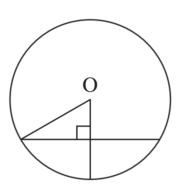
This creates a right-angled triangle, and Pythagoras' theorem or right-angled trigonometry can be applied, as demonstrated in the examples below.

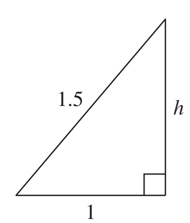
Example

Find the depth of oil in a pipe of diameter 3 m, shown below.



By drawing in radii as shown, the right-angled triangle is created:





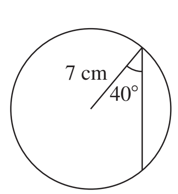
Applying Pythagoras' theorem:

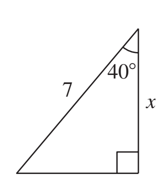
$$h^2 = 1.5^2 - 1^2$$
$$\Rightarrow h^2 = 1.25$$
$$\Rightarrow h = 1.118\ldots$$

So the depth is $3 - 1.118\ldots = 1.88$ m

Example

Find the length of the chord.



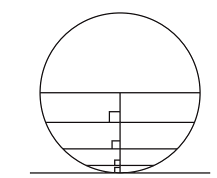


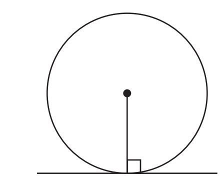
$$\cos 40° = \frac{x}{7}$$
$$\Rightarrow x = 7\cos 40°$$
$$\Rightarrow x = 5.36\ldots$$

Chord length $= 2 \times 5.36\ldots$
$= 10.7$ cm

Tangents

If we consider a chord moving downwards, the end points of the chord become closer together. When these end points become one point, the chord is a tangent.

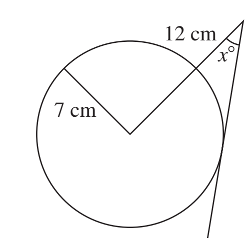




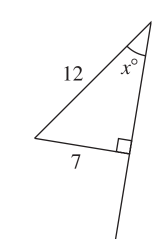
So a tangent is perpendicular to the radius at the point of contact. We can use this property to find angles.

Example

Find x.



Here we have the right-angled triangle



So $\sin x° = \frac{7}{12}$

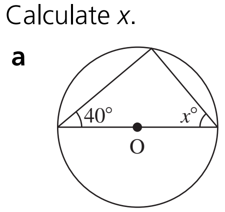
$$\Rightarrow x° = \sin^{-1}\left(\frac{7}{12}\right)$$

$$\Rightarrow x° = 35.7°$$

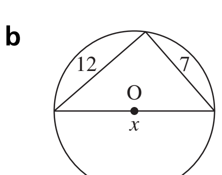
Exercise 3

1 Calculate x.

a

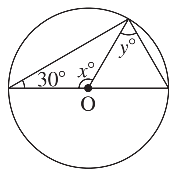


b

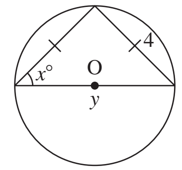


2 Calculate x and y.

a

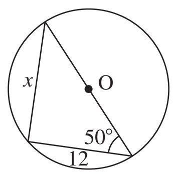


b

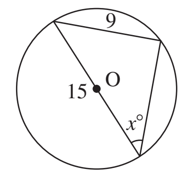


3 Calculate x.

a

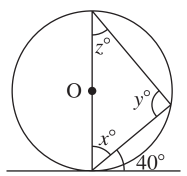


b



4 Find x, y and z.

a



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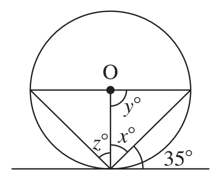
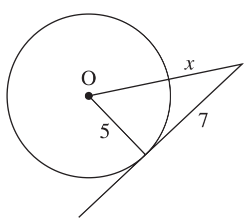
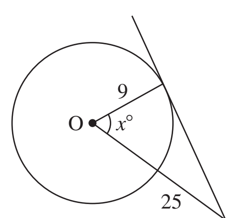


5 Calculate x.

a

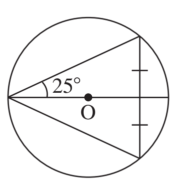


b

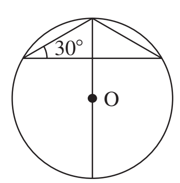


6 Copy these diagrams and fill in all of the angles.

a

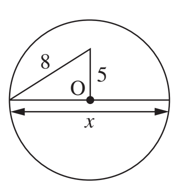


b

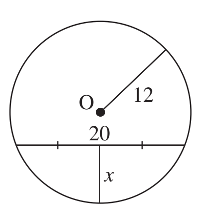


7 Calculate x.

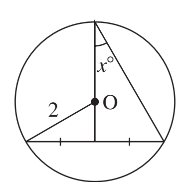
a



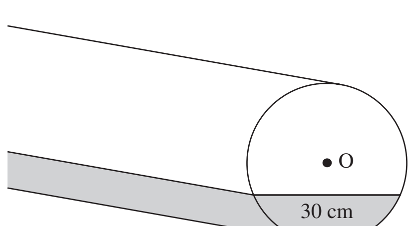
b



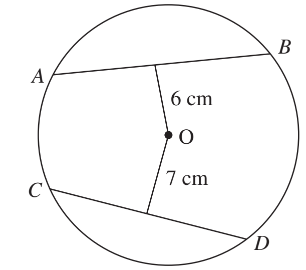
c



8 Calculate the depth of water in a pipe of diameter 50 cm, shown below.



9 The radius of the circle shown below is 12 cm.



Which chord is longer and by how much?

0.4 Surds and indices

Surds (radicals)

A surd is a number that is irrational (that is, it cannot be expressed as a fraction of two integers) and can be written $\sqrt{x}$, $x \in \mathbb{Z}$. There are properties and techniques that allow us to simplify surds.

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

Example

Simplify $\sqrt{48}$.

Here we look for the largest square number that is a factor of 48.

So $\sqrt{48} = \sqrt{16}\sqrt{3}$

$= 4\sqrt{3}$

Surds follow normal algebraic rules, as shown below.

Example

Simplify $\sqrt{18} - \sqrt{2} + \sqrt{8}$.

$$\begin{aligned}\sqrt{18} - \sqrt{2} + \sqrt{8} &= \sqrt{9}\sqrt{2} - \sqrt{2} + \sqrt{4}\sqrt{2} \\ &= 3\sqrt{2} - \sqrt{2} + 2\sqrt{2} \\ &= 4\sqrt{2}\end{aligned}$$

The distributive law can be applied to surds, as shown below.

Example

Expand $5(2\sqrt{3} + 7)$.

$5(2\sqrt{3} + 7) = 10\sqrt{3} + 35$

Example

Expand and simplify $(3+\sqrt{2})(2-\sqrt{2})$.

$$\begin{aligned}(3+\sqrt{2})(2-\sqrt{2}) &= 6-3\sqrt{2}+2\sqrt{2}-2\\ &= 4-\sqrt{2}\end{aligned}$$

Rationalizing the denominator

There is a mathematical convention that we do not generally have a surd on the denominator of a fraction. The process of making the denominator rational is called rationalizing the denominator. In order to rationalize the denominator, we multiply the numerator and denominator by a surd.

Example

Rationalize the denominator of $\frac{5}{\sqrt{2}}$.

Here we multiply numerator and denominator by $\sqrt{2}$.

$$\frac{5}{\sqrt{2}}=\frac{5}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{5\sqrt{2}}{2}$$

In order to rationalize the denominator of a fraction of the form $\frac{a}{p+\sqrt{q}}$, we multiply the numerator and denominator by the conjugate surd. The conjugate surd of $p+\sqrt{q}$ is $p-\sqrt{q}$. Similarly the conjugate surd of $\sqrt{p}+\sqrt{q}$ is $\sqrt{p}-\sqrt{q}$.

This is very similar to a difference of two squares, which will be covered in the algebra section of this chapter.

Example

Rationalize the denominator of $\frac{3}{4+\sqrt{3}}$.

$$\begin{aligned}\frac{3}{4+\sqrt{3}} &= \frac{3}{4+\sqrt{3}}\times\frac{4-\sqrt{3}}{4-\sqrt{3}}=\frac{3(4-\sqrt{3})}{(4+\sqrt{3})(4-\sqrt{3})}\\ &= \frac{12-3\sqrt{3}}{16-3}\\ &= \frac{12-3\sqrt{3}}{13}\end{aligned}$$

Example

Rationalize the denominator of $\frac{2}{\sqrt{6}-\sqrt{3}}$.

$$\begin{aligned}\frac{2}{\sqrt{6}-\sqrt{3}} &= \frac{2}{\sqrt{6}-\sqrt{3}}\times\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}+\sqrt{3}}=\frac{2(\sqrt{6}+\sqrt{3})}{\left(\sqrt{6}-\sqrt{3}\right)\left(\sqrt{6}+\sqrt{3}\right)}\\ &= \frac{2\sqrt{6}+2\sqrt{3}}{6-3}\\ &= \frac{2\sqrt{6}+2\sqrt{3}}{3}\end{aligned}$$

Indices

These rules help to simplify expressions that involve indices (or powers).

1. $x^p\times x^q=x^{p+q}$
2. $\frac{x^p}{x^q}=x^{p-q}$
3. $(x^p)^q=x^{pq}$
4. $x^{-p}=\frac{1}{x^p}$
5. $x^0=1$
6. $\sqrt[n]{x^m}=x^{\frac{m}{n}}$

Example

Simplify $\frac{x^5\times x^2}{x^3}$.

$$\frac{x^5\times x^2}{x^3}=\frac{x^7}{x^3}=x^4$$

Example

Express $2x^{-6}$ with a positive power.

$$2x^{-6}=\frac{2}{x^6}$$

Example

Simplify $\frac{3x^4\times 2x^{-2}}{5x^2}$.

$$\frac{3x^4\times 2x^{-2}}{5x^2}=\frac{6x^2}{5x^2}=\frac{6}{5}$$

Example

Simplify $\dfrac{3x^4}{\sqrt[3]{x^2}}$.

$$\frac{3x^4}{\sqrt[3]{x^2}} = \frac{3x^4}{x^{\frac{2}{3}}} = 3x^{\frac{10}{3}}$$

Example

Simplify $(2x^3)^4$.

$$(2x^3)^4 = 2^4x^{12} = 16x^{12}$$

Example

Simplify $\dfrac{5x^4 - 4x^7}{x^3}$.

$$\frac{5x^4 - 4x^7}{x^3} = x^{-3}(5x^4 - 4x^7)$$
$$= 5x - 4x^4$$

Example

Simplify $81^{\frac{3}{4}}$.

$$81^{\frac{3}{4}} = (81^{\frac{1}{4}})^3 = 3^3 = 27$$

Example

Simplify $\dfrac{6x^3 - 3x^{\frac{1}{2}}}{5\sqrt{x}}$.

$$\frac{6x^3 - 3x^{\frac{1}{2}}}{5\sqrt{x}} = \frac{6x^3 - 3x^{\frac{1}{2}}}{5x^{\frac{1}{2}}}$$
$$= \frac{1}{5}x^{-\frac{1}{2}}(6x^3 - 3x^{\frac{1}{2}})$$
$$= \frac{6}{5}x^{\frac{5}{2}} - \frac{3}{5}x^0$$
$$= \frac{6}{5}x^{\frac{5}{2}} - \frac{3}{5}$$

Exercise 4

1 Express these in simplest form.

a $\sqrt{20}$ **b** $\sqrt{32}$ **c** $\sqrt{63}$ **d** $\sqrt{44}$ **e** $\sqrt{300}$

2 Simplify these.

a $8\sqrt{5} - 3\sqrt{5}$ **b** $3\sqrt{2} - \sqrt{2} + 5\sqrt{2}$ **c** $8\sqrt{2} - 2\sqrt{8}$

d $6\sqrt{48} - \sqrt{12} + \sqrt{75}$ **e** $\sqrt{8} - \sqrt{32} + \sqrt{128}$

3 Simplify these.

a $\sqrt{3} \times \sqrt{6}$ **b** $\sqrt{8} \times \sqrt{12}$ **c** $\sqrt{10} \times \sqrt{20}$

d $3\sqrt{2} \times 5\sqrt{2}$ **e** $3\sqrt{5} \times 5\sqrt{3}$

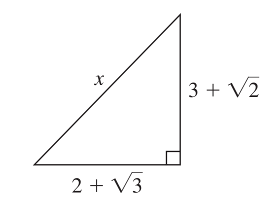
4 Simplify these.

a $4(1 + \sqrt{2})$ **b** $3(2 + \sqrt{3})$ **c** $4(3 - \sqrt{2})$

d $7(\sqrt{3} - 4)$ **e** $\sqrt{2}(3 + \sqrt{2})$ **f** $(3 + \sqrt{5})(2 + \sqrt{2})$

g $(2\sqrt{3} - 5)(4\sqrt{3} + 1)$ **h** $(2 + \sqrt{3})^2$ **i** $(5 - \sqrt{7})^2$

5 Calculate the exact value of x.



6 Rationalize the denominator of these fractions, simplifying where possible.

a $\dfrac{2}{\sqrt{3}}$ **b** $\dfrac{3}{\sqrt{5}}$ **c** $\dfrac{4}{\sqrt{2}}$ **d** $\dfrac{7}{\sqrt{18}}$

e $\dfrac{\sqrt{5}}{\sqrt{3}}$ **f** $\dfrac{10}{\sqrt{12}}$ **g** $\dfrac{2}{\sqrt{3} - 1}$ **h** $\dfrac{8}{\sqrt{5} + 1}$

i $\dfrac{12}{3 - \sqrt{2}}$ **j** $\dfrac{2}{\sqrt{3} - \sqrt{2}}$ **k** $\dfrac{1 - \sqrt{3}}{2 - \sqrt{5}}$ **l** $\dfrac{3 - \sqrt{5}}{2 + \sqrt{3}}$

7 Simplify each expression.

a $a^6 \times a^4$ **b** $k^{-4} \times k^7$ **c** $p^7 \times p$ **d** $z^{\frac{3}{2}} \times z^4$ **e** $\dfrac{x^6}{x^2}$

f $4k^5 \times 2k^{-3}$ **g** $8k^4 \div k^{-3}$ **h** $(x^3)^4$ **i** $(3p^2)^5$ **j** $(4k^{-3})^4$

k $(xy^2)^5$ **l** $(p^2q^3)^{-4}$ **m** $k^{\frac{1}{2}} \times k^{\frac{1}{4}}$ **n** $3x^{-\frac{1}{2}} \times 4x^{\frac{5}{2}}$

o $\dfrac{x^{\frac{3}{4}}}{x^{\frac{1}{4}}}$ **p** $4p^{\frac{1}{2}} \div 7p^{\frac{3}{2}}$ **q** $\dfrac{2x^5}{\sqrt[3]{x^5}}$

8 Simplify each expression, expressing your answer as a sum or difference of terms.

a $\dfrac{x^2 - 4x^3}{x^3}$ **b** $\dfrac{x^6 - 1}{x^2}$ **c** $\dfrac{x^7 + x^3}{2x^2}$ **d** $\dfrac{3x^2 + 4}{\sqrt{x}}$

e $\dfrac{8\sqrt[3]{x^4} - 3\sqrt[4]{x^5}}{\sqrt{x}}$ **f** $\dfrac{(x + 3)^2}{x}$ **g** $\dfrac{(x + 3)(x - 2)}{3x^2}$ **h** $\dfrac{(3 - 2x)^2}{4x}$

i $\left(\dfrac{5}{x} - 4\right)^2$ **j** $\dfrac{(x^2 - 3)(x + 1)}{x^2}$ **k** $\dfrac{(x + 2)^2}{\sqrt{x}}$

0.5 Expansion of brackets

To try to generalize work that can be done with numbers we use letters in place of numbers, and hence when we manipulate these, the mathematical operations that are used are identical to those for numbers, and the letters behave in exactly the same way.

We know that $2(3 + 4)$ is the same as $(2 \times 3) + (2 \times 4) = 6 + 8 = 14$.

Hence $x(2 + x)$ expands to $(2 \times x) + (x \times x) = 2x + x^2$.

Example

Expand and simplify $y(2 + y) - 2y(y - 3)$.

We begin by expanding each bracket separately:

$y(2 + y) - 2y(y - 3) = 2y + y^2 - 2y^2 + 6y$

Collecting together the 'like terms':

$y(2 + y) - 2y(y - 3) = -y^2 + 8y$

We normally write terms in decreasing order of power.

We often have to expand brackets when what is outside the bracket is not a single term, but another bracket. The next example demonstrates how this works with numbers.

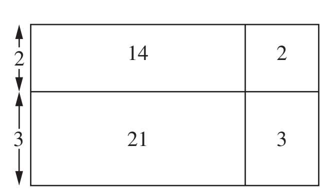
Example

Expand and simplify $(3 + 2)(7 + 1)$.

The simplest way of doing this is to simplify the brackets first and then to perform the multiplication:

$$(3 + 2)(7 + 1) = 5 \times 8 = 40$$

However, if we were to think of this as finding the area of a rectangle with one side 3 units + 2 units and the other side 7 units + 1 unit, we could work out the area by finding the area of four separate rectangles. This is shown on the diagram below.



Area $= 21 + 14 + 3 + 2 = 40$

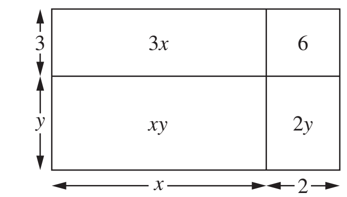
In this method we are multiplying each term in one bracket by each term in the other bracket:

$$\begin{aligned}(3 + 2)(7 + 1) &= (3)(7) + (2)(7) + (3)(1) + (2)(1)\\ &= 21 + 14 + 3 + 2 = 40\end{aligned}$$

We can use the rectangle method to expand brackets in this form containing algebra.

Example

Expand $(x + 2)(y + 3)$.

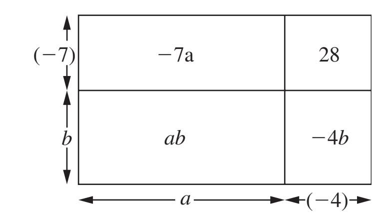


$$\begin{aligned}(x + 2)(y + 3) &= (x)(y) + (2)(y) + (x)(3) + (2)(3)\\ &= xy + 2y + 3x + 6\end{aligned}$$

Remember that the letters operate in exactly the same way as numbers.

Example

Expand $(a - 4)(b - 7)$.



Since this is actually the same as $(a + [-4])(b + [-7])$ we just keep the negative signs with their associated numbers.

$$\begin{aligned}(a - 4)(b - 7) &= (a)(b) + (-4)(b) + (a)(-7) + (-4)(-7)\\ &= ab - 4b - 7a + 28\end{aligned}$$

This is a rather long way of expanding brackets, and a quicker way could be to use the method called the "face method".

Example

Expand $(2m - 7)(3n + 9)$.

In this method we draw lines between the terms we need to multiply, then do the multiplications mentally. It is because the lines look like two eyes, a nose and a mouth that it can be called the "face method".

$$(2m - 7)(3n + 9) = 6mn - 21n + 18m - 63$$

If both brackets have an x-term, we can simplify the expanded expression.

Example

Expand and simplify $(x+2)(2x-5)$.

Using the face method: $(x+2)(2x-5) = 2x^2 + 4x - 5x - 10$

$= 2x^2 - x - 10$

This is known as a quadratic expression because the highest power in the expression is a 2.

If we are required to expand more than two terms in a bracket it works in exactly the same way.

Example

Expand and simplify $(2+3x-4y)(3x-4+9xy)$.

$(2+3x-4y)(3x-4+9xy) = 6x - 8 + 18xy + 9x^2 - 12x + 27x^2y$

$- 12xy + 16y - 36xy^2$

$= -6x - 8 + 6xy + 9x^2 + 27x^2y + 16y - 36xy^2$

Exercise 5

Expand and simplify where possible:

1 $2(4x-5)$
2 $3(y-1)+4(2y+5)$
3 $5(a-4)-3(a-9)$
4 $x(x-3)+2(2x-1)$
5 $2x(x-1)-x(5x+14)$
6 $3y(2y-1)-12(6-2y)$
7 $5(y-z)+15(z-3)+9z$
8 $a(2a-5)+5a(6a+9)-4a(a-12)$
9 $(x+2)(y+4)$
10 $(a-6)(b-7)$
11 $(2e+5)(3f-1)$
12 $(5s+12)(6t-17)$
13 $(x+5)(x+8)$
14 $(y-7)(y+8)$
15 $(y-7)(y+7)$
16 $(a-4)^2$
17 $(2x+5)(x-1)$
18 $(3b+2)(5b-3)$
19 $(1+5t)(4-3t)$
20 $(2s-1)(7-2s)$
21 $(2-3y)(3y+2)$
22 $(2x-9)^2$
23 $(3f-1)^2-2f(f-6)$
24 $(x+2y)^2-4xy$
25 $5(2x-1)(x+3)$
26 $(a+3)^2-(2a+1)^2$
27 $16-(4+r)^2$
28 $(x+y+2)(x-y-4)$
29 $(2x+y-5)(x+xy)$
30 $(a+b+c)(3-a)$
31 $(3x+5y-7)(2x-7y+2)$
32 $2(a-2b+7)(a-ab+5)$
33 $6(x-y-3z)(x+4y-z)$

0.6 Factorisation

Often in mathematics we need to undo a process, and the "undoing" of expansion is factorisation or factoring. Again this works in the same way as with numbers. If we consider 15, the factors of 15 are 3 and 5 since $3 \times 5 = 15$. Consider the expression $5y^2 - 10y$. The factors of this are 5, y and $(y-2)$. Hence $5y^2 - 10y = 5y(y-2)$. To factorise expressions like this we usually look for factors that are common to all terms and put these in front of the bracket.

Example

Factorise $16x^2 - 4x$.

From this we can see that the factors that are common to both terms are 4 and x. Since $16x^2 \div 4x = 4x$ and $-4x \div 4x = -1$, $16x^2 - 4x = 4x(4x-1)$.

Example

Factorise $9a^2b - 27ab + 3ab^2$.

We note that the factors that are common to all three terms are 3, a and b. Since $9a^2b \div 3ab = 3a$, $-27ab \div 3ab = -3$ and $3ab^2 \div 3ab = b$, $9a^2b - 27ab + 3ab^2 = 3ab(3a - 9 + b)$.

Sometimes the factor may not be just a single number or letter, and it may be necessary to factorise in two stages.

Example

Factorise $3ac - 6bc + 4ad - 8bd$.

We can see that the first two terms have a common factor of $3c$ and the last two terms have a common factor of $4d$.
Hence $3ac - 6bc + 4ad - 8bd = 3c(a-2b) + 4d(a-2b)$.
Since $a - 2b$ is now a common factor,
$3ac - 6bc + 4ad - 8bd = (a-2b)(3c+4d)$.

It is now necessary to see how this works with quadratic expressions. Not all quadratic expressions will factorise with integers. From the work done on expansion using the "face method" we saw that one "eye" produces the x^2-term, the other "eye" produces the constant term, and the addition or subtraction of the "nose" and "mouth" produces the x-term. It is now a matter of undoing this.

Example

Factorise $x^2 + 9x + 8$.

To produce x^2 the only possibilities are x and x.
To produce 8, the possible factors are 8 and 1 or 4 and 2. Since it is not possible to produce $+9$ from 4 and 2 by addition or subtraction, it must be 8 and 1.
Since we are trying to produce $+8$, it must be either $+8$ and $+1$ or -8 and -1. As we need $+9x$, then it must be $+8$ and $+1$.
Hence $x^2 + 9x + 8 = (x + 8)(x + 1)$.
We should always check the answer is correct by expanding the brackets:
$(x + 8)(x + 1) = x^2 + 8x + x + 8 = x^2 + 9x + 8$

Sometimes two factors appear to work, and we need to distinguish which ones are the correct ones.

Example

Factorise $x^2 - 5x + 6$.

To produce x^2 the only possibilities are x and x.
To produce 6, the possible factors are 6 and 1 or 3 and 2. In this example it is possible to produce 5 from 6 and 1 by subtraction or from 3 and 2 by addition. However, we are trying to produce $+6$, and therefore the signs of the two numbers must either both be positive or both be negative. As we need -5, then it must be -3 and -2.
Hence $x^2 - 5x + 6 = (x - 3)(x - 2)$.
Check the answer is correct by expanding the brackets:
$(x - 3)(x - 2) = x^2 - 3x - 2x + 6 = x^2 - 5x + 6$

Once this has been practised it should be possible to write the answers straight down, without showing any working.

It is also important to remember a special case. We saw from the previous exercise that brackets of the form $(x - a)(x + a)$ expand to $x^2 - a^2$, since the x-terms cancel out, and this is known as the difference of two squares. This should be remembered.

$$x^2 - a^2 = (x - a)(x + a)$$

Example

Factorise $x^2 - 49$.

Since $\sqrt{49} = \pm 7$, $x^2 - 49 = (x - 7)(x + 7)$.

Sometimes the coefficient of x^2 is not 1, and in this case it makes the factorisation more complicated. However, if there is a common factor in the expression, then this may simplify the matter.

Example

Factorise $4x^2 - 4x - 48$.

We can see that 4 is a common factor and hence $4x^2 - 4x - 48 = 4(x^2 - x - 12)$.
We now factorise $x^2 - x - 12$ in the usual way.
To produce x^2 the only possibilities are x and x.
To produce 12, the possible factors are 12 and 1 , 6 and 2, or 3 and 4. As we want to produce -1 by addition or subtraction it must be 3 and 4.
Since we are trying to produce -12, it must be either -3 and $+4$ or $+3$ and -4. As we need $-1x$, then it must be $+3$ and -4.
Hence $x^2 - x - 12 = (x - 4)(x + 3)$.
Check the answer is correct by expanding the brackets:
$(x - 4)(x + 3) = x^2 - 4x + 3x - 12 = x^2 - x - 12$
Hence $4x^2 - 4x - 48 = 4(x - 4)(x + 3)$.

Sometimes there is no common factor to remove, and we need to deal with the fact that the coefficient of x^2 is not 1.

Example

Factorise $2x^2 + 9x + 10$.

To produce $2x^2$ the only possibilities are $2x$ and x.
To produce 10, the possibilities are 1 and 10 or 2 and 5. As we want to produce $+10$, it must be either -2 and -5 or $+2$ and $+5$. As we need $+9x$, it must be $+2$ and $+5$.
Hence $2x^2 + 9x + 10 = (2x + 5)(x + 2)$.

Check the answer is correct by expanding the brackets:
$(2x + 5)(x + 2) = 2x^2 + 5x + 4x + 10 = 2x^2 + 9x + 10$

Example

Factorise $4x^2 - 5x - 6$.

To produce $4x^2$ the possibilities are $2x$ and $2x$ or x and $4x$.
To produce 6, the possibilities are 2 and 3 or 1 and 6. As we want to produce -6, the signs of each pair must be opposite.
To determine which pairs we use, we set up a table that shows the possible arrangements.

A	B	C	D	$AB + CD$
$2x$	-2	$2x$	3	$2x$
x	1	$4x$	-6	$-23x$
x	-6	$4x$	1	$-2x$
x	-1	$4x$	6	$23x$
x	6	$4x$	-1	$2x$
x	-3	$4x$	2	$5x$
x	2	$4x$	-3	$-10x$
x	3	$4x$	-2	$-5x$
x	-2	$4x$	3	$10x$

From this we can see that we need x and $4x$ and 3 and -2 since this gives $-5x$.
Hence $4x^2 - 5x - 6 = (4x + 3)(x - 2)$.
Check the answer is correct by expanding the brackets:
$(4x + 3)(x - 2) = 4x^2 + 3x - 8x - 6 = 4x^2 - 5x - 6$

Exercise 6

Factorise the following:

1 $3x - 15$ **2** $5x^2 - 2x$
3 $3a + 15b$ **4** $2xy^2 + 12x^2y$
5 $10xy + 15y - 5x^3y$ **6** $ac + 2bc + 2cd + 4bd$
7 $2ac - 6bc + ad - 3bd$ **8** $4xy - 8y^2 - xz + 2yz$
9 $12ey + 8fy - 9eh - 6fh$ **10** $7sy - 5tx - 35ty + sx$
11 $x^2 + 3x + 2$ **12** $x^2 + 6x + 5$ **13** $x^2 + 5x - 14$
14 $x^2 - 9x - 22$ **15** $x^2 - 12x + 35$ **16** $x^2 + 5x + 6$
17 $x^2 + 8x + 12$ **18** $x^2 - 7x - 8$ **19** $x^2 + x - 12$
20 $2x^2 + 10x + 12$ **21** $3x^2 - 3x - 36$ **22** $x^2 - 9x + 18$
23 $2x^2 + 5x + 2$ **24** $2x^2 + 13x + 20$ **25** $3x^2 + 5x - 2$
26 $5x^2 + 6x - 8$ **27** $3x^2 + 10x - 25$ **28** $4x^2 + 8x + 3$
29 $6x^2 + 17x + 5$ **30** $4x^2 - 31x - 8$ **31** $12x^2 - 7x - 12$
32 $28x^2 - 41x + 15$

0.7 Algebraic fractions

These behave in exactly the same way as numerical fractions. Hence if we are multiplying or dividing we need to cancel common factors, and if we are adding or subtracting we need to put the fractions over a common denominator.

Example

Simplify $\frac{12a^2}{2b} \times \frac{9ab}{3a}$.

$$\frac{\overset{6}{\cancel{12}}a^{\cancel{2}}}{2\cancel{b}} \times \frac{\overset{3}{\cancel{9}}a\cancel{b}}{\cancel{3a}} = 18a^2$$

Example

Simplify $\frac{3x^2y}{12x^3} \div \frac{15x^2y^2}{4y}$.

$$\frac{3x^2y}{12x^3} \div \frac{15x^2y^2}{4y} = \frac{\cancel{3x^2y}}{\underset{3}{\cancel{12}}x^3} \times \frac{\cancel{4y}}{\underset{5}{\cancel{15}}\cancel{x^2y^2}} = \frac{1}{15x^3}$$

Change the division to a multiplication, as with numerical fractions.

Sometimes it is necessary to factorise the numerator and/or the denominator before simplification can take place.

Example

Simplify $\frac{x^2 - x - 12}{x^2 - 6x - 8}$.

Factorising the numerator and the denominator:

$$\frac{x^2 - x - 12}{x^2 - 6x + 8} = \frac{\cancel{(x - 4)}(x + 3)}{\cancel{(x - 4)}(x - 2)} = \frac{x + 3}{x - 2}$$

Example

Simplify $\frac{3}{2x} + \frac{5}{x^2}$.

Since we are adding two fractions we need to find the common denominator of the two fractions. This is $2x^2$.

$$\frac{3}{2x} + \frac{5}{x^2} = \frac{3x}{2x^2} + \frac{10}{2x^2} = \frac{3x + 10}{2x^2}$$

Example

Simplify $\frac{x + 2}{4} - \frac{x - 1}{3}$.

In this case the common denominator is 12.

$$\frac{x + 2}{4} - \frac{x - 1}{3} = \frac{3(x + 2)}{12} - \frac{4(x - 1)}{12}$$
$$= \frac{3x + 6 - 4x + 4}{12}$$
$$= \frac{10 - x}{12}$$

Example

Simplify $\dfrac{2}{f+g} - \dfrac{f-g}{2}$.

In this example the common denominator is $2(f+g)$.

$$\frac{2}{f+g} - \frac{f-g}{2} = \frac{4}{2(f+g)} - \frac{(f+g)(f-g)}{2(f+g)}$$
$$= \frac{4-f^2+g^2}{2(f+g)}$$

Exercise 7

Simplify the following, if possible:

1 $\dfrac{x-3}{4x-12}$ **2** $\dfrac{5a+20}{3a+12}$ **3** $\dfrac{x^3+x^2y}{xy+y^3}$

4 $\dfrac{a-6}{2a-3}$ **5** $\dfrac{(x+5)(x-6)}{x^2-25}$ **6** $\dfrac{(x-3)(x+2)}{x^2+x-2}$

7 $\dfrac{12p+36q}{p^2+6pq+9q^2}$ **8** $\dfrac{8x^2}{y} \times \dfrac{3x}{4y}$ **9** $5s^4t^3 \times \dfrac{2t}{s^3}$

10 $x(3+2x) \div \dfrac{3+2x}{4}$ **11** $\dfrac{4\pi r^3}{3} \div 2\pi r^2$ **12** $\dfrac{3}{4x^2-1} \div \dfrac{6}{2x-1}$

13 $(3x+1) \times \dfrac{5}{9x^2-1}$ **14** $\dfrac{5a^4}{3} \times \dfrac{7a^2}{30}$ **15** $\dfrac{a^2b}{c} \div \dfrac{bc}{a^2}$

16 $\dfrac{x^2-x-6}{9} \div (x-3)$ **17** $\dfrac{2}{f} - \dfrac{3}{g}$ **18** $z + \dfrac{1}{z}$

19 $\dfrac{1}{4}(2x+1) + \dfrac{1}{3}(3x-2)$ **20** $\dfrac{2}{\sin A} + \dfrac{3}{\sin B}$ **21** $7y + \dfrac{1}{28y}$

22 $\dfrac{z}{a^2} + \dfrac{z}{b^2}$ **23** $\dfrac{1}{y+1} + \dfrac{1}{y-1}$ **24** $\dfrac{6}{x+4} + \dfrac{3}{x+3}$

25 $\dfrac{7}{a^2-4} - \dfrac{2}{a-2}$ **26** $\dfrac{1}{4z^2+4z+1} - \dfrac{3}{2z+1}$

27 $\dfrac{9}{(2x+1)^2} + \dfrac{3}{2x+1}$ **28** $\dfrac{5}{4(y+3)} - \dfrac{4}{3(3y-2)}$

29 $\dfrac{1}{x-1} + \dfrac{4}{x+5} - \dfrac{3}{x-2}$ **30** $\dfrac{5}{y^2-x^2} - \dfrac{3}{y-x}$

31 $\dfrac{7t}{t^2-4t-5} + \dfrac{6}{t+1}$ **32** $\dfrac{1}{x-1} + \dfrac{2x}{1-x^2}$ **33** $\dfrac{2x}{x^2+x-6} + \dfrac{2x}{x-2}$

34 $\dfrac{7}{x^2+3x-10} - \dfrac{2}{x^2+5x} - \dfrac{2}{x^2-2x}$ **35** $\dfrac{2+x}{x+3} - \dfrac{x-2}{x-3}$

36 $\dfrac{3x-1}{3x^2+11x+6} + \dfrac{2x}{x+3}$ **37** $\dfrac{2x}{x^2+5xy+6y^2} + \dfrac{2x}{x+3y}$

38 $\dfrac{2x+4}{x^2+x-6} \times \dfrac{x^2-9}{12x+24} \div \dfrac{3x^2+3x}{(2x^2+4x)^2}$

0.8 Linear equations

With linear equations the technique is to ensure we have all the terms in the unknown on one side and all the constant terms on the other. We do this using the operations of $+, -, \times$ and $\div$. Since there is an equals sign in an equation, provided we perform the same mathematical operation on each side, then the equation remains valid.

Example

Solve $4x - 15 = 15x + 7$.

To separate the terms in x and the constant terms we subtract $15x$ from both sides and add 15 to both sides.

$$4x - 15 = 15x + 7$$
$$\Rightarrow 4x - 15x - 15 + 15 = 15x - 15x + 7 + 15$$
$$\Rightarrow -11x = 22$$
$$\Rightarrow \frac{-11x}{-11} = \frac{22}{-11}$$
$$\Rightarrow x = -2$$

Example

Solve $3(x-1) - 5(2x-3) = 4(3x+4)$.

In this case we begin by expanding out the brackets.

$$3(x-1) - 5(2x-3) = 4(3x+4)$$
$$\Rightarrow 3x - 3 - 10x + 15 = 12x + 16$$
$$\Rightarrow -7x + 12 = 12x + 16$$

Collecting like terms

To separate the terms in x and the constant terms we subtract $12x$ from both sides and subtract 12 from both sides.

$$\Rightarrow -7x - 12x + 12 - 12 = 12x - 12x + 16 - 12$$
$$\Rightarrow -19x = 4$$
$$\Rightarrow \frac{-19x}{-19} = \frac{4}{-19}$$
$$\Rightarrow x = -\frac{4}{19}$$

Example

Solve $\dfrac{3(x+1)}{2} - \dfrac{2(2x-3)}{5} = 4$.

In this situation we begin by putting the fraction over a common denominator.

$$\frac{3(x+1)}{2} - \frac{2(2x-3)}{5} = 4$$
$$\Rightarrow \frac{15(x+1)}{10} - \frac{4(2x-3)}{10} = 4$$

We now multiply both sides of the equation by 10 to remove the denominator.

$$\begin{aligned} &\Rightarrow 15(x+1) - 4(2x-3) = 40 \\ &\Rightarrow 15x + 15 - 8x + 12 = 40 \\ &\Rightarrow 7x + 27 = 40 \\ &\Rightarrow 7x + 27 - 27 = 40 - 27 \\ &\Rightarrow 7x = 13 \\ &\Rightarrow \frac{7x}{7} = \frac{13}{7} \\ &\Rightarrow x = \frac{13}{7} \end{aligned}$$

Expanding out the brackets

Collecting like terms

Subtracting 27 from both sides

Example

Solve $\frac{5}{2x+1} = \frac{7}{3x-2}$.

In this case we begin by putting both sides of the equation over a common denominator.

$$\begin{aligned} &\frac{5}{2x+1} = \frac{7}{3x-2} \\ &\Rightarrow \frac{5(3x-2)}{(2x+1)(3x-2)} = \frac{7(2x+1)}{(2x+1)(3x-2)} \\ &\Rightarrow 5(3x-2) = 7(2x+1) \\ &\Rightarrow 15x - 10 = 14x + 7 \end{aligned}$$

This is what we call cross multiplication and could have been applied directly.

Expanding out the brackets

To separate the terms in x and the constant terms we subtract $14x$ from both sides and add 7 to both sides.

$$\begin{aligned} &\Rightarrow 15x - 14x - 10 + 10 = 14x - 14x + 7 + 10 \\ &\Rightarrow x = 17 \end{aligned}$$

Exercise 8

Solve these equations:

1 $4x - 15 = 3x + 9$

2 $5x + 2 - 3x - 4 = 6x - 7$

3 $6x + 3 - 4x - 9 = 5x - 2 + 3x$

4 $\frac{x}{2} + 3 = 5 - \frac{x}{3}$

5 $\frac{x}{35} + 15 = 21$

6 $-\frac{x}{12} + 2 = -\frac{1}{3}$

7 $3(x-1) + 2 = 4x$

8 $2x - 1 - 3(x-1) = 2(x-2)$

9 $12y - (y-5) = 21$

10 $3(a-2) + 2(2a+1) = 5$

11 $2(2b-5) - (2b+6) = 3(2b-1)$

12 $(7-t) - (t-4) - (-3t-5) = \frac{t}{2}$

13 $2(2p+1)^2 - 4(2p^2-3) = -3(p+1)$

14 $\frac{x}{2} + \frac{2x}{3} = \frac{1}{6}$

15 $\frac{y}{5} + \frac{2}{3} - \frac{1}{30} = -\frac{2}{15}$

16 $\frac{a+1}{4} + \frac{a-2}{2} = \frac{3}{16}$

17 $\frac{2(x-1)}{7} + \frac{7(x+1)}{2} = 3$

18 $\frac{2x-3}{4} + \frac{3x-4}{7} = \frac{3}{14}$

19 $\frac{3(2x-5)}{5} - \frac{4(3x-2)}{3} = \frac{5}{2}$

20 $\frac{x+1}{3} = \frac{x-2}{4}$

21. $\frac{14}{x-3} = \frac{10}{x+5}$

22. $\frac{2x+5}{4} = \frac{3x-1}{5}$

23. $\frac{2(x-3)}{5} = \frac{3(2x-1)}{7}$

24. $\frac{3(2x-5)}{7} = -\frac{2(2x+1)}{5}$

0.9 Rearranging mathematical formulae

This works in a very similar way to solving linear equations, with the aim of putting the subject of the formula on one side of the equation on its own. Again we do this by using standard operations of adding, subtracting, multiplying and dividing.

Example

Make x the subject of the formula $fx + 2g = gx - 5f$.

Since we want x to be the subject of the formula we gather the terms in x on one side of the equation and all other terms on the other side. Hence we subtract gx and $2g$ from both sides of the equation.

$$\begin{aligned} &fx + 2g = gx - 5f \\ &\Rightarrow fx - gx + 2g - 2g = gx - gx - 5f - 2g \\ &\Rightarrow fx - gx = -5f - 2g \end{aligned}$$

To get the x-term on its own we now take x out as a common factor.

$$\begin{aligned} &\Rightarrow x(f-g) = -5f - 2g \\ &\Rightarrow x = \frac{-5f-2g}{f-g} \end{aligned}$$

Dividing both sides by $(f-g)$

Since both terms in the numerator are negative we can say $x = \frac{-(5f+2g)}{f-g}$.

Example

Make y the subject of the formula $\frac{y}{a+2} - \frac{y+1}{3a} = 3$.

In this case we begin by putting the fractions over a common denominator.

$$\Rightarrow \frac{3ay}{3a(a+2)} - \frac{(a+2)(y+1)}{3a(a+2)} = 3$$

We now multiply both sides by the common denominator.

$$\begin{aligned} &\Rightarrow 3ay - (a+2)(y+1) = 9a(a+2) \\ &\Rightarrow 3ay - ay - a - 2y - 2 = 9a^2 + 18a \\ &\Rightarrow 2ay - a - 2y - 2 = 9a^2 + 18a \end{aligned}$$

Expanding the brackets

Since we want y to be the subject of the formula we gather the terms in y on one side of the equation and all other terms on the other side. Hence we add a and 2 to both sides of the equation.

$$\Rightarrow 2ay - a + a - 2y - 2 + 2 = 9a^2 + 18a + a + 2$$
$$\Rightarrow 2ay - 2y = 9a^2 + 19a + 2$$

To get the y term on its own we now take y out as a common factor.

$$\Rightarrow y(2a - 2) = 9a^2 + 19a + 2$$
$$\Rightarrow y = \frac{9a^2 + 19a + 2}{(2a - 2)}$$

Dividing both sides by $(2a - 2)$

Since 2 is a common factor of the denominator, we can say $y = \dfrac{9a^2 + 19a + 2}{2(a - 1)}$.

Example

Make z the subject of the formula $\dfrac{c}{2z + 1} = \dfrac{2c - 1}{4z - 3}$.

In this case we remove the fractions by multiplying both sides by $(2z + 1)(4z - 3)$.

$$\Rightarrow \frac{c(2z + 1)(4z - 3)}{2z + 1} = \frac{(2c - 1)(2z + 1)(4z - 3)}{4z - 3}$$
$$\Rightarrow c(4z - 3) = (2c - 1)(2z + 1)$$

With practice, we can go straight to this line.

$$\Rightarrow 4cz - 3c = 4cz + 2c - 2z - 1$$

Expanding the brackets

Since we want z to be the subject of the formula we gather the terms in z on one side of the equation and all other terms on the other side. Hence we subtract $4cz$ and $2c$ from both sides of the equation and add 1 to both sides of the equation.

$$\Rightarrow 4cz - 4cz - 3c - 2c + 1 = 4cz - 4cz + 2c - 2c - 2z + 1 - 1$$
$$\Rightarrow -5c + 1 = -2z$$
$$\Rightarrow z = \frac{-5c + 1}{-2}$$

Dividing both sides by -2

$$\Rightarrow z = \frac{5c - 1}{2}$$

Multiplying numerator and denominator of the fraction by -1

Example

Make t the subject of the formula $s = pq - qt^2$.

Since we want t to be the subject of the formula we need to get the term containing t on its own. Hence we subtract pq from both sides of the equation.

$$\Rightarrow s - pq = pq - pq - qt^2$$
$$\Rightarrow s - pq = -qt^2$$

It is easier if the term containing t is positive, so we multiply both sides of the equation by -1.

$$\Rightarrow qt^2 = pq - s$$
$$\Rightarrow t^2 = \frac{pq - s}{q}$$

Dividing both sides by q

To get t, we need the square root of t^2 and hence we take the square root of both sides of the equation.

$$t = \pm\sqrt{\frac{pq - s}{q}}$$

Exercise 9

In the following questions, make the letter given in brackets the subject of the formula.

1 $x = 2y - 3z$ (z)

2 $3y + bc = 2a$ (c)

3 $2y + 5z = 3y - x$ (y)

4 $ab - 2ac = 3c^2$ (a)

5 $a(x - 1) + b(2x + 3) = 3x$ (x)

6 $\dfrac{y}{2z} + \dfrac{y}{2x} = 2$ (y)

7 $\dfrac{f + g}{h} + \dfrac{2f}{h} = k$ (f)

8 $\dfrac{z + 1}{a} + \dfrac{z - 2}{b} = 5$ (z)

9 $\dfrac{b}{3a + 2} - \dfrac{b + 1}{5a} = 9$ (b)

10 $\dfrac{b}{3a + 2} - \dfrac{b + 1}{5} = 9$ (a)

11 $\dfrac{x}{a} = \dfrac{a}{b - 1}$ (x)

12 $\dfrac{m - n}{p} = \dfrac{m - p}{n}$ (m)

13 $\dfrac{2y}{3z - 1} = \dfrac{2y - 5}{4z - 7}$ (y)

14 $\dfrac{2y}{3z - 1} = \dfrac{2y - 5}{4z - 7}$ (z)

15 $\dfrac{ax^2 - bx}{c} = \dfrac{2a + 3x}{4c - 1}$ (a)

16 $p = qt^2 + 3$ (t)

17 $a = 3b^2 - 2a^2$ (b)

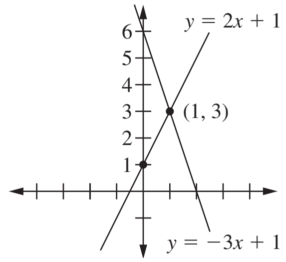
18 $\dfrac{s}{3t^2 - 1} = \dfrac{3s^3}{2s - 1}$ (t)

19 $\dfrac{a}{3b^2 - 1} + \dfrac{2}{3a - 1} = 5$ (b)

20 $\dfrac{a}{3b^2 - 1} + \dfrac{2}{3a - 1} = 5$ (b)

0.10 Simultaneous equations

This is the scenario when we have two equations in two unknowns and we need to find the values of the two unknowns that fit both equations. We have already met an equation in two unknowns which we drew as a straight-line graph. Graphically, when we are solving two equations in two unknowns we are finding the point of intersection of the two lines. If we draw $y = 2x + 1$ and $y = -3x + 6$ we can see that the point of intersection of the two lines is (1, 3) and hence the solution to the equations $y = 2x + 1$ and $y = -3x + 6$ is $x = 1$ and $y = 3$.



However, drawing a graph is not an accurate way of finding the solution and therefore we use one of two alternative methods – elimination or substitution.

Elimination

In this technique we eliminate one of the letters from the equations. This is done by adding or subtracting the equations.

Example

Solve $\begin{aligned}3x - 2y &= 2\\ 5x + 2y &= 6\end{aligned}$ using a method of elimination.

We begin by labelling the equations.

$3x - 2y = 2$ equation (i)
$5x + 2y = 6$ equation (ii)

We see here that if we add the two equations together, then this will eliminate *y*.

$\Rightarrow 8x = 8$ (i) + (ii)
$\Rightarrow x = 1$

To find *y* we now substitute back in either of the equations. Using equation (i):

$\Rightarrow 3 - 2y = 2$
$\Rightarrow -2y = -1$
$\Rightarrow y = \frac{1}{2}$

We can check this is correct by substituting in equation (ii):

$5(1) + 2\left(\frac{1}{2}\right) = 6$
$\Rightarrow 5 + 1 = 6$ which is true.

Sometimes we have to modify the equations before we add or subtract them.

Example

Solve $\begin{aligned}2x + y &= 7\\ 5x + 3y &= 16\end{aligned}$ using a method of elimination.

Again we begin by labelling the equations.

$2x + y = 7$ equation (i)
$5x + 3y = 16$ equation (ii)

In this case we see that neither adding nor subtracting the equations will eliminate either *x* or *y*.
However, if we multiply equation (i) by 3, then we can eliminate *y*.
Equation (i) $\times 3 \Rightarrow 6x + 3y = 21$ equation (iii)
Subtracting equation (ii) from equation (iii) $\Rightarrow x = 5$
To find y we now substitute back in any of the equations. Using equation (i):

$\Rightarrow 10 + y = 7$
$\Rightarrow y = -3$

We can check this is correct by substituting in equation (ii):

$5(5) + 3(-3) = 16$
$\Rightarrow 25 - 9 = 16$ which is true.

In some cases we need to modify both equations before we add or subtract them.

Example

Solve $\begin{aligned}6x + 7y &= 12\\ 4x + 3y &= 18\end{aligned}$ using a method of elimination.

Again we begin by labelling the equations.

$6x + 7y = 12$ equation (i)
$4x + 3y = 18$ equation (ii)

In this case it does not matter which letter we eliminate first as both cases involve modifying both equations. We will begin by eliminating *x*.
Hence we multiply equation (i) by 4 and equation (ii) by 6.
$\Rightarrow 24x + 28y = 48$ equation (iii)
$\Rightarrow 24x + 18y = 108$ equation (iv)
We now subtract equation (iv) from equation (iii):

$\Rightarrow 10y = -60$
$\Rightarrow y = -6$

To find *x* we now substitute back in any of the equations. Using equation (i):

$\Rightarrow 6x - 42 = 12$
$\Rightarrow 6x = 54$
$\Rightarrow x = 9$

We can check this is correct by substituting in equation (ii):

$\Rightarrow 4(9) + 3(-6) = 18$
$\Rightarrow 36 - 18 = 18$ which is true.

Substitution

In this case we rearrange one of the equations to find one letter in terms of the other and then substitute in the second equation.

Example

Solve $\begin{aligned} a + 2b &= 9 \\ 3a - 4b &= 7 \end{aligned}$ using a method of substitution.

As before we begin by labelling the equations.

$a + 2b = 9$ equation (i)

$3a - 4b = 7$ equation (ii)

From equation (i) $a = 9 - 2b$

Substituting in equation (ii) $\Rightarrow 3(9 - 2b) - 4b = 7$

$\Rightarrow 27 - 6b - 4b = 7$

$\Rightarrow 27 - 10b = 7$

$\Rightarrow -10b = -20$

$\Rightarrow b = 2$

We know that $a = 9 - 2b$

$\Rightarrow a = 9 - 2(2)$

$\Rightarrow a = 5$

Again we can check by substituting in equation (ii):

$\Rightarrow 3(5) - 4(2) = 7$

$\Rightarrow 15 - 8 = 7$ which is true.

Example

Solve $\begin{aligned} 3x - 2y &= 11 \\ 2x + 5y &= 1 \end{aligned}$ using a method of substitution.

As before we begin by labelling the equations.

$3x - 2y = 11$ equation (i)

$2x + 5y = 1$ equation (ii)

In this case it does not matter which letter we substitute. From equation (ii):

$2x = 1 - 5y$

$\Rightarrow x = \dfrac{1 - 5y}{2}$

Substituting in equation (i) $\Rightarrow 3\left(\dfrac{1 - 5y}{2}\right) - 2y = 11$

Multiplying throughout by $2 \Rightarrow 3(1 - 5y) - 4y = 22$

$\Rightarrow 3 - 15y - 4y = 22$

$\Rightarrow 3 - 19y = 22$

$\Rightarrow -19y = 19$

$\Rightarrow y = -1$

We know that $x = \dfrac{1 - 5y}{2}$

$\Rightarrow x = \dfrac{1 - 5(-1)}{2}$

$\Rightarrow x = \dfrac{6}{2} = 3$

Again we can check by substituting in equation (i):

$\Rightarrow 3(3) - 2(-1) = 11$

$\Rightarrow 9 + 2 = 11$ which is true.

Exercise 10

Solve the following simultaneous equations using a method of elimination.

1 $x + 2y = 5$, $x - 3y = -5$

2 $3x + y = 11$, $x + y = 7$

3 $4x + 5y = -1$, $3x - 5y = 8$

4 $3x + 7y = -13$, $x - 7y = 5$

5 $3x + 2y = 7$, $2x + y = 4$

6 $x - 3y = -5$, $3x + y = 15$

7 $x - 2y = 5$, $5x + 3y = -14$

8 $3x + 2y = 12$, $2x + 5y = 19$

9 $3x - 5y = -1$, $7x + 3y = 5$

10 $3x + 7y + 23 = 0$, $y = 2x + 14$

Solve the following simultaneous equations using a method of substitution.

11 $x + y = 5$, $2x - y = -2$

12 $2x + y = 9$, $3x - 2y = -4$

13 $5x - y = 8$, $3x + 2y = -3$

14 $6x - 4y = 0$, $2x + y = -7$

15 $x + 5y = 7$, $5x - 3y = -7$

16 $2x + 3y = 23$, $3x + 2y = 17$

17 $5x - 2y = 9$, $6x + 7y = -8$

18 $2x - 7y = 19$, $3x - 9y = 24$

19 $3x + 4y = -19$, $4x - 9y = 32$

20 $5x - 7y + 1 = 0$, $2y + 5x - 17 = 0$

0.11 Probability

Below is a series of introductory practicals that can be undertaken.

A. Heads or tails

Equipment

A coin
A sheet of paper to record the results
A calculator

Method

1. Flip the coin.
2. Record the result each time using H or T.
3. Repeat 20 times.
4. Count the number of heads.
5. Write this as a fraction of the total number of trials and convert to a decimal.
6. Flip the coin another 20 times and count the number of heads.
7. Add this to the previous number of heads, express as a fraction out of 40 and again convert to a decimal.
8. Repeat this after every 20 flips until the coin has been flipped 200 times.

B. Six or not a six

Equipment

An unbiased die
A sheet of paper to record the results
A calculator

Method

1. Throw the die.
2. Record the result each time using 6 or $\overline{6}$.
3. Repeat 20 times.
4. Count the number of sixes.
5. Write this as a fraction of the total number of trials and convert to a decimal.
6. Throw the die another 20 times and count the number of sixes.
7. Add this to the previous number of sixes, express as a fraction out of 40 and again convert to a decimal.
8. Repeat this after every 20 throws until the die has been thrown 200 times.

The notation $\overline{6}$ is standard notation for "not a 6".

C. Red or blue

Equipment

8 red counters and 4 blue counters
A bag
A sheet of paper to record the results
A calculator

Method

1. Place the counters in the bag.
2. Pick a counter from the bag, note its colour as R or B, and replace the counter in the bag.
3. Repeat 20 times.
4. Count the number of red counters.
5. Write this as a fraction of the total number of trials and convert to a decimal.
6. Pick another 20 counters and count the number of red counters.
7. Add this to the previous number of red counters, express as a fraction out of 40 and again convert to a decimal.
8. Repeat this after every 20 counters until 200 counters have been picked.

Results

From any of these experiments you should find that the proportions initially fluctuate, but as you approach 200 trials they should start to settle down. If you were to continue beyond 200, there would be even less fluctuation. In practical A, the decimal should be about 0.5, in practical B it should be about 0.16 and in practical C it should be about 0.33. What these experiments have given is a relative frequency, and this relative frequency seems to be approaching a limiting value in each case. This limiting value is known as the **probability**.