20 Probability

Have you ever asked around a group of your classmates and been surprised to find that two of them share the same birthday? At first thought it seems likely that for two people in the room to share a birthday there will need to be a lot of people in the room. However, this is not actually the case. If we want to have more than a 50% chance of two people in the room having the same birthday then we can calculate, using probability, that the number required is 23 or more. However, if we change the problem to having 50% chance of finding someone in the room with the same birthday as you, then the number required increases dramatically – approximately 254 people are needed. Hence it should come as no surprise that two presidents of the United States of America have shared the same birthday and three presidents have died on the same day!
The way in which we can investigate problems like this is by using probability, but we will begin with much simpler problems.

20.1 Introduction to probability

In the presumed knowledge section (Chapter 0 – see accompanying CD) we considered the idea that when we look at an experimental situation we find answers that indicate that a theoretical application is appropriate. This theoretical approach is called probability and is what we will explore in this chapter. Consider a number of equally likely outcomes of an event. What is the probability of one specific outcome of that event? For example, if we have a cubical die what is the probability of throwing a six? Since there are six equally likely outcomes and only one of them is throwing a six, then the probability of throwing a six is 1 in 6. We would normally write this as a fraction $\frac{1}{6}$ or as a decimal or a percentage. Since probability is a theoretical concept, it does not mean that if we throw a die six times we will definitely get a six on one of the throws. However, as the number of trials increases, the number of sixes becomes closer to $\frac{1}{6}$ of the total.
Generally, if the probability space S consists of a finite number of equally likely outcomes, then the probability of an event E, written $\mathrm{P}(E)$ is defined as:

$\mathrm{P}(E) = \frac{\mathrm{n}(E)}{\mathrm{n}(S)}$ where $\mathrm{n}(E)$ is the number of occurrences of the event E and $\mathrm{n}(S)$ is the total number of possible outcomes.

Hence in a room of fifteen people, if seven of them have blue eyes, then the probability that a person picked at random will have blue eyes is $\frac{7}{15}$.

Important results

$$0 \le \mathrm{P}(A) \le 1$$

If an event A can never happen, the probability is 0, and if it will certainly happen, the probability is 1. This can be seen from the fact that $\mathrm{P}(A) = \frac{\mathrm{n}(A)}{\mathrm{n}(S)}$ since if the event can never happen then $\mathrm{n}(A) = 0$ and if the event is certain to happen then $\mathrm{n}(A) = \mathrm{n}(S)$. Since $\mathrm{n}(S)$ is always greater than or equal to $\mathrm{n}(A)$, a probability can never be greater than 1.

A' is known as the complement of A.

$\mathrm{P}(A) + \mathrm{P}(A') = 1$ where $\mathrm{P}(A')$ is the probability that the event A does not occur.

Example

A bag contains a large number of tiles. The probability that a tile drawn from the bag shows the letter A is $\frac{3}{10}$, the probability that a tile drawn from the bag shows the letter B is $\frac{5}{10}$, and the probability that a tile drawn from the bag shows the letter C is $\frac{2}{10}$.

What is the probability that a tile drawn at random from the bag

a shows the letter A or B

b does not show the letter C?

a Since the probability of showing the letter A is $\frac{3}{10}$ and the probability of showing the letter B is $\frac{5}{10}$, the probability of showing the letter A or B is $\frac{3}{10} + \frac{5}{10} = \frac{8}{10} = \frac{4}{5}$.

b The probability of not showing the letter C can be done in two ways. We can use the complement and state that

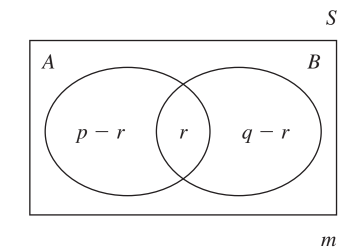
$$\mathrm{P}(\text{not } C) = \mathrm{P}(C') = 1 - \mathrm{P}(C) = 1 - \frac{2}{10} = \frac{4}{5}.$$

Alternatively, we can see that this is the same as showing a letter A or B and hence is the same as the answer to part **a**.

$$\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \text{ and } B)$$

or using set notation: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

In this case the two intersecting sets A and B represent the events A and B and the universal set represents the sample space S. This is shown in the Venn diagram. Venn diagrams and set notation are introduced in the presumed knowledge chapter (Chapter 0 – see accompanying CD).



Proof

If $\mathrm{n}(S) = m$, $\mathrm{n}(A) = p$, $\mathrm{n}(B) = q$ and $\mathrm{n}(A \cap B) = r$ then

$$\begin{aligned}\mathrm{P}(A \cup B) &= \frac{\mathrm{n}(A \cup B)}{\mathrm{n}(S)} \\ &= \frac{(p - r) + r + (q - r)}{m} \\ &= \frac{p}{m} + \frac{q}{m} - \frac{r}{m} \\ &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\end{aligned}$$

Example

A tetrahedral die and a cubical die are thrown. What is the probability of throwing a five on the cubical die or a four on the tetrahedral die?

We could use the formula above directly, but we will demonstrate what is happening here by using a possibility space diagram that shows all the possible outcomes of the event "throwing a cubical die and a tetrahedral die".

4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	1	2	3	4	5	6

From the diagram we see that the probability of throwing a five on the cubical die is $\frac{4}{24}$.

The probability of throwing a four on the tetrahedral die is $\frac{6}{24}$.

Hence at first it may appear that the probability of throwing a five on the cubical die or a four on the tetrahedral die is $\frac{6}{24} + \frac{4}{24}$. However, from the diagram we notice that the occurrence four on the tetrahedral die and five on the cubical die appears in both calculations. Hence we need to subtract this probability. Therefore the probability of throwing a five on the cubical die or a four on the tetrahedral die is $\frac{6}{24} + \frac{4}{24} - \frac{1}{24} = \frac{9}{24}$.

Example

In a group of 20 students, there are 12 girls and 8 boys. Two of the boys and three of the girls wear red shirts. What is the probability that a person chosen randomly from the group is either a boy or someone who wears a red shirt?

Let A be the event "being a boy" and B be the event "wearing a red shirt".

Hence $\mathrm{P}(A) = \frac{8}{20}$, $\mathrm{P}(B) = \frac{5}{20}$ and $\mathrm{P}(A \cap B) = \frac{2}{20}$.

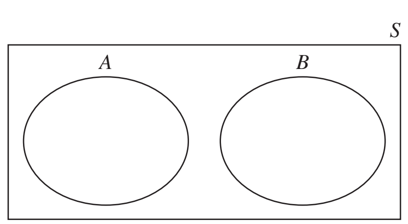
Now

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$$\Rightarrow \mathrm{P}(A \cup B) = \frac{8}{20} + \frac{5}{20} - \frac{2}{20} = \frac{11}{20}$$

If an event A can occur or an event B can occur but A and B cannot both occur, then the two events A and B are said to be mutually exclusive.
In this case $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A \cap B) = 0$.

We can see this from the Venn diagram, where there is no overlap between the sets.



For mutually exclusive events $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

Example

A bag contains 3 red balls, 4 black balls and 3 yellow balls. What is the probability of drawing either a red ball or a black ball from the bag?

Let the event "drawing a red ball" be A and the event "drawing a black ball" be B.

$\mathrm{P}(A) = \frac{3}{10}$ and $\mathrm{P}(B) = \frac{4}{10}$

Since these are mutually exclusive events

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

$$\Rightarrow \mathrm{P}(A \cup B) = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$$

If two events A and B are such that $A \cup B = S$, where S is the total probability space, then $\mathrm{P}(A \cup B) = 1$ and the events A and B are said to be exhaustive.

Example

The events A and B are exhaustive. If $\mathrm{P}(A) = 0.65$ and $\mathrm{P}(B) = 0.44$, find $\mathrm{P}(A \cap B)$.

We know that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

Since the events are exhaustive $\mathrm{P}(A \cup B) = 1$

$\Rightarrow 1 = 0.65 + 0.44 - \mathrm{P}(A \cap B)$

$\Rightarrow \mathrm{P}(A \cap B) = 0.09$

Exercise 1

1 An unbiased tetrahedral die is thrown. What is the probability of throwing
- **a** a three
- **b** an even number
- **c** a prime number?

2 A spinner has the numbers 1 to 10 written on it. When spun, it is equally likely to stop on any of the ten numbers. What is the probability that it will stop on
- **a** a three
- **b** an odd number
- **c** a multiple of 3
- **d** a prime number?

3 A bag contains 5 black balls, 6 white balls, 7 pink balls and 2 blue balls. What is the probability that a ball drawn randomly from the bag will
- **a** be a black ball
- **b** be either a white or a pink ball
- **c** not be a blue ball
- **d** be either a black, white or blue ball
- **e** be a red ball?

4 Sheila is picking books off her shelf. The shelf only contains mathematics books and novels. The probability that she picks a novel is 0.48 and the probability that she picks a mathematics book is 0.52.
- **a** What is the probability that she does not pick a mathematics book?
- **b** Explain why these events are exhaustive.

5 If $\mathrm{P}(A) = \frac{1}{5}$, $\mathrm{P}(B) = \frac{2}{3}$ and $\mathrm{P}(A \cap B) = \frac{4}{15}$, are A and B exhaustive events?

6 The probability that Hanine goes to the local shop is $\frac{3}{7}$. The probability that she does not cycle is $\frac{4}{11}$. The probability that she goes to the shop and cycles is $\frac{4}{15}$.
- **a** What is the probability that she cycles?
- **b** What is the probability that she cycles or goes to the shop?

7 It is given that for two events A and B, $P(A) = \frac{3}{8}$, $P(A \cup B) = \frac{11}{16}$ and $P(A \cap B) = \frac{3}{16}$. Find $P(B)$.

8 In a class, 6 students have brown eyes, 3 students have blue eyes, 4 students have grey eyes and 2 students have hazel eyes. A student is chosen at random. Find the probability that

a a student with blue eyes is chosen
b a student with either blue or brown eyes is chosen
c a student who does not have hazel eyes is chosen
d a student with blue, brown or grey eyes is chosen
e a student with grey or brown eyes is chosen.

9 Two tetrahedral dice are thrown. What is the probability that

a the sum of the two scores is 5
b the sum of the two scores is greater than 4
c the difference between the two scores is 3
d the difference between the two scores is less than 4
e the product of the two scores is an even number
f the product of the two scores is greater than or equal to 6
g one die shows a 3 and the other die shows a number greater than 4?

10 Two cubical dice are thrown. What is the probability that

a the sum of the two scores is 9
b the sum of the two scores is greater than 4
c the difference between the two scores is 3
d the difference between the two scores is at least 4
e the product of the two scores is 12
f the product of the two scores is an odd number
g one die shows an even number or the other die shows a multiple of 3?

11 The probability that John passes his mathematics examination is 0.9, and the probability that he passes his history examination is 0.6. These events are exhaustive. What is the probability that

a he does not pass his mathematics examination
b he passes his history examination or his mathematics examination
c he passes his mathematics examination and his history examination?

12 In a school's IB diploma programme, 30 students take at least one science. If 15 students take physics and 18 students take chemistry, find the probability that a student chosen at random studies both physics and chemistry.

13 There are 20 students in a class. In a class survey on pets, it is found that 12 students have a dog, 5 students have a dog and a rabbit and 3 students do not have a dog or a rabbit. Find the probability that a student chosen at random will have a rabbit.

14 In a survey of people living in a village, all respondents either shop at supermarket A, supermarket B or both. It is found that the probability that a person will shop at supermarket A is 0.65 and the probability that he/she will shop at supermarket B is 0.63. If the probability that a person shops at both supermarkets is 0.28, find the probability that a person from the village chosen at random will shop at supermarket A or supermarket B, but not both.

15 Two cubical dice are thrown. What is the probability that the sum of the two scores is

a a multiple of 3
b greater than 5
c a multiple of 3 and greater than 5
d a multiple of 3 or greater than 5

e less than 4 or one die shows a 5?
f Explain why the events in part **e** are mutually exclusive.

16 A class contains 15 boys and 17 girls. Of these 10 boys and 8 girls have blonde hair. Find the probability that a student chosen at random is a boy or has blonde hair.

17 Two tetrahedral dice are thrown. What is the probability that the sum of the scores is

a even
b prime
c even or prime?
d Explain why these two events are mutually exclusive.

18 When David goes fishing the probability of him catching a fish of type A is 0.45, catching a fish of type B is 0.75 and catching a fish of type C is 0.2. David catches four fish.
If the event X is David catching two fish of type A and two other fish, the event Y is David catching two fish of type A and two of type B and the event Z is David catching at least one fish of type C, for each of the pairs of X, Y and Z state whether the two events are mutually exclusive, giving a reason.

19 If A and B are exhaustive events, and $P(A) = 0.78$ and $P(B) = 0.37$, find $P(A \cap B)$.

20 A whole number is chosen from the numbers 1 to 500. Find the probability that the whole number is

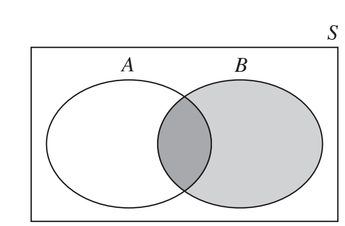
a a multiple of 6
b a multiple of both 6 and 8.

20.2 Conditional probability

If A and B are two events, then the probability of A given that B has already occurred is written as $P(A|B)$. This is known as **conditional probability**.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

On the Venn diagram below the possibility space is the set B as this has already occurred.



Hence $P(A|B) = \dfrac{n(A \cap B)}{n(B)}$

If all the possible events are represented by the universal set S, then

$$P(A|B) = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

A card is picked at random from a pack of 20 cards numbered 1, 2, 3, …, 20. Given that the card shows an even number, find the probability that it is a multiple of 4.

Let the event A be "picking a card showing a multiple of 4" and let the event B be "picking a card showing an even number".

Hence we require $P(A|B) = \frac{P(A \cap B)}{P(B)}$

In this case $P(A \cap B) = \frac{5}{20}$ and $P(B) = \frac{10}{20}$

$\Rightarrow P(A|B) = \frac{\frac{5}{20}}{\frac{10}{20}} = \frac{1}{2}$

Alternatively, we could write the result as $P(A \cap B) = P(A|B) \times P(B)$.

Example

Two tetrahedral dice are thrown; one is red and the other is blue. The faces are marked 1, 2, 3, 4. Given that the red die lands on an odd number, the probability that the sum of the scores on the dice is 6 is $\frac{1}{8}$. Find the probability that the sum of the scores on the dice is 6 and the red die lands on an odd number.

Let A be the event "the sum of the scores is 6" and let B be the event "the red die lands on an odd number".

We know $P(A|B) = \frac{1}{8}$ and $P(B) = \frac{1}{2}$

Hence $P(A \cap B) = P(A|B) \times P(B) = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$

If A and B are mutually exclusive events then since $P(A \cap B) = 0$ and $P(B) \neq 0$, it follows that $P(A|B) = 0$.

Exercise 2

1 For two events A and B it is given that $P(A) = \frac{5}{18}$, $P(B) = \frac{5}{9}$ and $P(A|B) = \frac{3}{14}$. Find

   a $P(A \cap B)$ b $P(B|A)$

2 A bag contains 6 balls, each with a number between 4 and 9 written on it. Each ball has a different number written on it. Find the probability that if two balls are drawn

   a the sum of the scores is greater than 12

   b the second ball shows a 7, given that the sum of the scores is greater than 12

   c the first ball is even, given that the difference between the numbers is 3.

3 Two tetrahedral dice are thrown. Find the probability that

   a at least one of the dice shows a 3

   b the difference between the scores on the two dice is 2

   c given that at least one of the dice shows a 3, the difference between the scores on the dice is 2

   d given that the difference between the scores on the dice is 2, the product of the scores on the dice is 8.

4 In a game of Scrabble, Dalene has the seven letters A, D, E, K, O, Q and S. She picks two of these letters at random.

   a What is the probability that one is a vowel and the other is the letter D?

   b If the first letter she picks is a consonant, what is the probability that the second letter is the E?

   c Given that she picks the letter Q first, what is the probability that she picks the letter D or the letter K second?

5 There are ten discs in a bag. Each disc has a number on it between 0 and 9. Each number only appears once. Hamish picks two discs at random. Given that the first disc drawn shows a multiple of 4, what is the probability that

   a the sum of the numbers on the two discs is less than10

   b the sum of the numbers on the two discs is even

   c the difference between the two numbers on the discs is less than 3?

6 On any given day in June the probability of it raining is 0.24. The probability of Suzanne cycling to work given that it is raining is 0.32. Find the probability that Suzanne cycles to work and it is raining.

7 Events A and B are such that $P(A) = \frac{4}{13}$ and $P(B) = \frac{9}{16}$. The conditional probability $P(A|B) = 0$.

   a Find $P(A \cup B)$.

   b Are A and B exhaustive events? Give a reason for the answer.

8 The probability of Nick gaining a first class degree at university given that he does 25 hours revision per week is 0.7. The probability that he gains a first class degree and does 25 hours revision per week is 0.85. Find the probability that he does 25 hours revision.

9 A team of two is to be picked from Alan, Bruce, Charlie and Danni.

   a Draw a possibility space diagram to show the possible teams of two.

   b What is the probability that if Danni is chosen, either Alan or Bruce will be her partner?

   c Given that Alan or Bruce are chosen, what is the probability that Danni will be the other person?

20.3 Independent events

If the occurrence or non-occurrence of an event A does not influence in any way the probability of an event B then the event B is said to be **independent** of event A.

In this case $P(B|A) = P(B)$.

Now we know that $P(A \cap B) = P(B|A)P(A)$

$\Rightarrow P(A \cap B) = P(B) \times P(A) = P(A) \times P(B)$

This is only true if A and B are independent events.

For independent events, $P(A \text{ and } B) = P(A) \times P(B)$.

To tackle questions on independent events we sometimes use a possibility space diagram but a more powerful tool is a **tree diagram**.

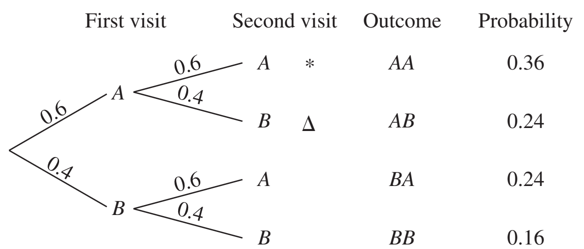
Example

A man visits his local supermarket twice in a week. The probability that he pays by credit card is 0.4 and the probability that he pays with cash is 0.6. Find the probability that

a he pays cash on both visits

b he pays cash on the first visit and by credit card on the second visit.

We will use A to mean paying by cash and B to mean paying with a credit card. The tree diagram is shown below.



The probabilities of his method of payment are written on the branches.
By multiplying the probabilities along one branch we find the probability of one outcome. Hence in this situation there are four possible outcomes. If we add all the probabilities of the outcomes together, the answer will be 1.

a In this case we need the branch marked *.

$P(AA) = 0.6 \times 0.6 = 0.36$

P(AA) means paying by cash on the first visit and on the second visit.

b In this case we need the branch marked Δ.

$P(AB) = 0.6 \times 0.4 = 0.24$

Example

A bag contains 3 red sticks, 5 white sticks and 2 blue sticks. A stick is taken from the bag, the colour noted then replaced in the bag. Another stick is then taken. Find the probability that

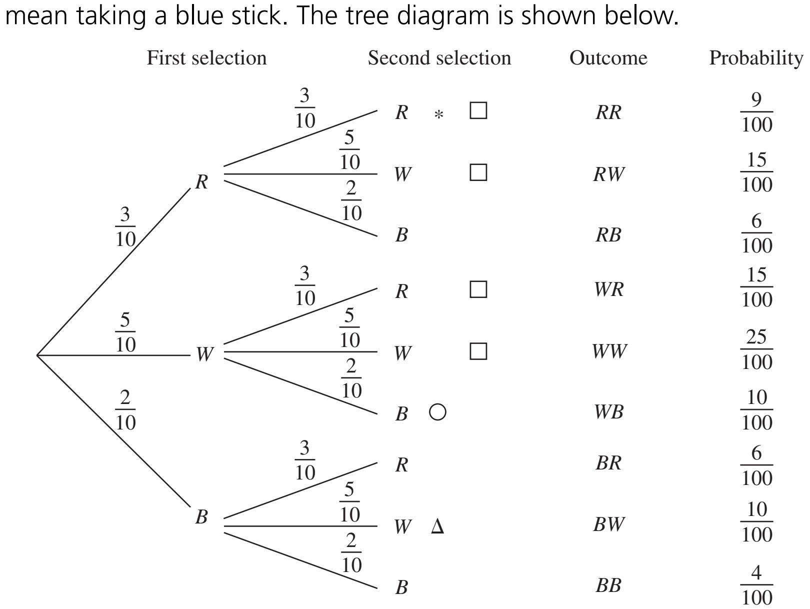
a both sticks are red

b a blue stick is drawn first and then a white stick

c one blue stick and one white stick are taken

d at least one stick is blue.

We will use R to mean taking a red stick, W to mean taking a white stick and B to mean taking a blue stick. The tree diagram is shown below.



The probability of taking a certain colour of stick is written on the appropriate branch.
By multiplying the probabilities along one branch we find the probability of one outcome. Hence there are nine possible outcomes. If we add all the probabilities of the outcomes together, the answer will be 1.

a We need the branch marked *.

$$P(RR) = \frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$$

P(RR) means the probability of taking a red stick and then taking another red stick.

b We need the branch marked Δ.

$$P(BW) = \frac{2}{10} \times \frac{5}{10} = \frac{10}{100} = \frac{1}{10}$$

c The order in which we take the blue stick and the white stick does not matter. Thus we require the probability of taking a blue stick followed by a white stick and the probability of taking a white stick followed by a blue stick. Hence we use two separate branches and then add the answers. The two branches required are marked Δ and ○.

$$P(BW) + P(WB) = \left(\frac{2}{10} \times \frac{5}{10}\right) + \left(\frac{5}{10} \times \frac{2}{10}\right) = \frac{20}{100} = \frac{1}{5}$$

d We could add all the branches that contain an event B. Since we know the total probability is 1, it is actually easier to subtract the probabilities of those branches that do not contain an event B from 1. These are marked □.

$$P(\text{at least one blue}) = 1 - \{P(RR) + P(WW) + P(RW) + P(WR)\}$$

$$\Rightarrow P(\text{at least one blue}) = 1 - \left\{\left(\frac{3}{10} \times \frac{3}{10}\right) + \left(\frac{5}{10} \times \frac{5}{10}\right) + \left(\frac{3}{10} \times \frac{5}{10}\right) + \left(\frac{5}{10} \times \frac{3}{10}\right)\right\}$$

$$\Rightarrow P(\text{at least one blue}) = \frac{26}{100} = \frac{13}{50}$$

This is an example of sampling with replacement, which means that the probabilities do not change from one event to the subsequent event. Below is an example of sampling with replacement where they do change.

Example

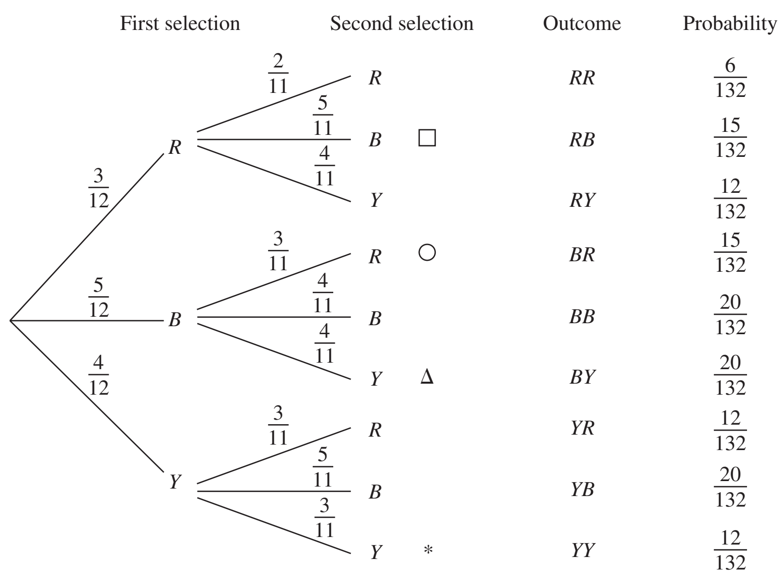
A box contains 3 red balls, 5 blue balls and 4 yellow balls. Keith draws a ball from the box, notes its colour and discards it. He then draws another ball from the box and again notes the colour. Find the probability that

a both balls are yellow

b Keith draws a blue ball the first time and a yellow ball the second time

c Keith draws a red ball and a blue ball.

We will use R to mean drawing a red ball, B to mean drawing a blue ball and Y to mean drawing a yellow ball. The tree diagram is shown below.



a We need the branch marked *.

$$P(YY) = \frac{4}{12} \times \frac{3}{11} = \frac{1}{11}$$

b We need the branch marked Δ.

$$P(BY) = \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$$

c The order in which Keith takes the red ball and the blue ball does not matter. We require the probability of taking a red ball followed by a blue ball and the probability of taking a blue ball followed by a red. Hence we use two separate branches and then add the answers. The two branches required are marked □ and ○.

$$P(RB) + P(BR) = \left(\frac{3}{12} \times \frac{5}{11}\right) + \left(\frac{5}{12} \times \frac{3}{11}\right) = \frac{30}{132} = \frac{5}{22}$$

We can also use tree diagrams to help us with conditional probability.

Example

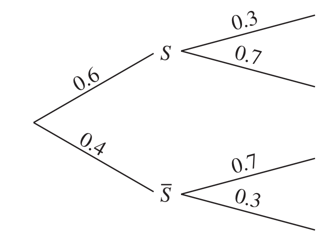
In the school canteen 60% of students have salad as a starter. Of the students who have salad as a starter, 30% will have cheesecake as dessert. Of those who do not have salad as a starter, 70% will have cheesecake as dessert.

a Show this information on a tree diagram.

b Given that Levi chooses to have salad as a starter, what is the probability that he will choose cheesecake as dessert?

c Given that Levi does not have cheesecake as dessert, what is the probability that he chose salad as a starter?

a We will use S to mean having salad and C to mean having cheesecake.



b We want to find $P(C|S)$. We could use the formula here, but it can be seen directly from the tree diagram that $P(C|S) = 0.3$.

By the formula

$$P(C|S) = \frac{P(C \cap S)}{P(S)} = \frac{P(S \cap C)}{P(S)}$$

$$= \frac{0.6 \times 0.3}{0.6} = 0.3$$

c Here we want $P(S|C)$ and in this situation it is easier to use the formula.

$$P(S|C) = \frac{P(S \cap C)}{P(C)}$$

$$= \frac{0.6 \times 0.3}{(0.6 \times 0.3) + (0.4 \times 0.7)} = \frac{9}{23}$$

Exercise 3

1 In a mathematics test the probability that Aly scores more than 70% is 0.6. In a physics test the probability that he scores more than 70% is 0.5. What is the probability that

a he scores more than 70% in both tests

b he scores more than 70% in only one test?

2 A cubical die is thrown twice.

a Draw a tree diagram to show the outcomes "throwing a three" and "not throwing a three".

b What is the probability that both dice show a three?

c What is the probability that neither dice shows a three?

d Draw a tree diagram to show the outcomes "throwing a number less than four" and "throwing a number greater than or equal to four".

e What is the probability that only one die shows a number less than four?

f What is the probability that at least one die shows a number less than four?

3 On any particular day, the probability that it rains is 0.2. The probability that a soccer team will win is 0.6 if it is raining and 0.7 if it is not raining. The team plays once in a week.

a Draw a tree diagram to show these events and their outcomes.

b What is the probability that it will rain and the team will win?

c What is the probability that the team will lose?

d Given that it is not raining, what is the probability that they will lose?

e Given that they win, what is the probability that it was raining?

4 Three fair coins are tossed. Each coin can either land on a head or a tail. What is the probability of gaining

a three heads

b two heads and a tail

c a tail on the first toss followed by a head or a tail in either order on the second toss

d at least one tail?

5 Two fair coins are tossed. Each coin can either land on a head or a tail.

a Show the possible outcomes on a tree diagram.

b What is the probability of getting at least one head?

A third coin is now tossed which is twice as likely to show heads as tails.

c Add an extra set of branches to the tree diagram to show the possible outcomes.

d What is the probability of getting two heads and a tail?

e What is the probability of getting two tails and a head, given that the third coin lands on tails?

6 The letters of the word PROBABILITY are placed in a bag. A letter is selected, it is noted whether it is a vowel or a consonant, and returned to the bag. A second letter is then selected and the same distinction is noted.

a Draw a tree diagram to show the possible outcomes.

b What is the probability of noting two consonants?

c What is the probability of noting a vowel and a consonant?

7 A box contains 4 blue balls, 3 red balls and 5 green balls. Three balls are drawn from the box without replacement. What is the probability that

a all three balls are green

b one ball of each colour is drawn

c at least one blue ball is drawn

d a pink ball is drawn

e no red balls are drawn?

f Given that the second ball is blue, what is the probability that the other two are either both red, both green, or one each of red or green?

8 Bag A contains 6 blue counters and 4 green counters. Bag B contains 9 blue counters and 5 green counters. A counter is drawn at random from bag A and two counters are drawn at random from bag B. The counters are not replaced.

a Find the probability that the counters are all blue.

b Find the probability that the counters are all the same colour.

c Given that there are two blue counters and one green counter, what is the probability that the green counter was drawn from bag B?

9 Events A and B are such that $\mathrm{P}(B) = \frac{2}{5}$, $\mathrm{P}(A|B) = \frac{1}{3}$ and $\mathrm{P}(A \cup B) = \frac{4}{5}$.

a Find $\mathrm{P}(A \cap B)$.

b Find $\mathrm{P}(A)$.

c Show that A and B are not independent.

10 Six cards a placed face down on a table. Each card has a single letter on it. The six letters on the cards are B, H, K, O, T and U. Cards are taken from the table and not replaced. Given that the first card drawn shows a vowel, what is the probability that the second card shows

a the letter B

b one of the first ten letters of the alphabet

c the letter T or the letter K?

11 Zahra catches the train to school every day from Monday to Friday. The probability that the train is late on a Monday is 0.35. The probability that it is late on any other day is 0.42. A day is chosen at random. Given that the train is late that day, what is the probability that the day is Monday?

12 Jane and John are playing a game with a biased cubical die. The probability that the die lands on any even number is twice that of the die landing on any odd number. The probability that the die lands on an even number is $\frac{2}{9}$. If the die shows a 1, 2, 3 or 4 the player who threw the die wins the game. If the die shows a 5 or a 6 the other player has the next throw. Jane plays first and the game continues until there is a winner.

a What is the probability that Jane wins on her first throw?

b What is the probability that John wins on his first throw?

c Calculate the probability that Jane wins the game.

13 The events A and B are independent. If $\mathrm{P}(A) = 0.4$ and $\mathrm{P}(B|A) = 0.2$, find

a $\mathrm{P}(B)$

b the probability that A occurs or B occurs, but not both A and B.

14 Janet has gone shopping to buy a new dress. To keep herself entertained whilst shopping she is listening to her iPod, which she takes off when she tries on a new dress. The probability that she leaves her iPod in the shop is 0.08. After visiting two shops in succession she finds she has left her iPod in one of them. What is the probability that she left her iPod in the first shop?

15 A school selects three students at random from a shortlist of ten students to be prefects. There are six boys and four girls.

a What is the probability that no girl is selected?

b Find the probability that two boys and one girl are selected.

16 A and B are two independent events. $\mathrm{P}(A) = 0.25$ and $\mathrm{P}(B) = 0.12$. Find

a $\mathrm{P}(A \cap B)$ **b** $\mathrm{P}(A \cup B)$ **c** $\mathrm{P}(A|B)$ **d** $\mathrm{P}(B|A)$

17 A soccer player finds that when the weather is calm, the probability of him striking his target is 0.95. When the weather is windy, the probability of him striking his target is 0.65. According to the local weather forecast, the probability that any particular day is windy is 0.45.

a Find the probability of him hitting the target on any randomly chosen day.

b Given that he fails to hit the target, what is the probability that the day is calm?

20.4 Bayes' theorem

We begin with the result

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A|B) \times \mathrm{P}(B)}{\mathrm{P}(A)}$$

Proof

We know $\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \Rightarrow \mathrm{P}(A \cap B) = \mathrm{P}(A|B)\mathrm{P}(B)$

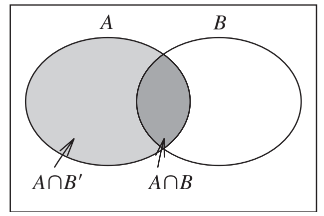
and $\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} \Rightarrow \mathrm{P}(A \cap B) = \mathrm{P}(B|A)\mathrm{P}(A)$

So $\mathrm{P}(A|B)\mathrm{P}(B) = \mathrm{P}(B|A)\mathrm{P}(A)$

$$\Rightarrow \mathrm{P}(B|A) = \frac{\mathrm{P}(A|B) \times \mathrm{P}(B)}{\mathrm{P}(A)}$$

This result can be written in a different form.

From the Venn diagram we see that



$$\begin{aligned}P(A) &= P(A\cap B) + P(A\cap B')\\ &= P(B)\frac{P(A\cap B)}{P(B)} + P(B')\frac{P(A\cap B')}{P(B')}\\ &= P(B)P(A|B) + P(B')P(A|B')\end{aligned}$$

Substituting into the formula for $P(B|A)$ gives:

$$P(B|A) = \frac{P(A|B)\times P(B)}{P(B)P(A|B) + P(B')P(A|B')}$$

This is also a useful result to remember.

This is known as **Bayes' theorem for two events** and is used if we are given $P(A|B)$ and need $P(B|A)$.

Example

A town has only two bus routes. Route A has twice as many buses as route B. The probability of a bus running late on route A is $\frac{1}{8}$ and the probability of it running late on route B is $\frac{1}{10}$. At a certain point on the route the buses run down the same road. If a passenger standing at the bus stop sees a bus running late, use Bayes' theorem to find the probability that it is a route B bus.

Let the probability of a route B bus be $P(B)$ and the probability of a bus being late be $P(L)$.

Since there are only two bus routes the probability of a route A bus is $P(B')$.

Bayes' theorem states $P(B|L) = \dfrac{P(L|B)\times P(B)}{P(B)P(L|B) + P(B')P(L|B')}$.

We are given that $P(B) = \frac{1}{3}$, $P(B') = \frac{2}{3}$, $P(L|B) = \frac{1}{10}$ and $P(L|B') = \frac{1}{8}$.

$$\Rightarrow P(B|L) = \frac{\frac{1}{10}\times\frac{1}{3}}{\left(\frac{1}{3}\times\frac{1}{10}\right) + \left(\frac{2}{3}\times\frac{1}{8}\right)} = \frac{2}{7}$$

This question could have been done using a tree diagram and conditional probability.

"At least" problems

We have already met the idea of the total probability being one and that sometimes it is easier to find a probability by subtracting the answer from one. In some situations we do not have a choice in this, as shown in the example below.

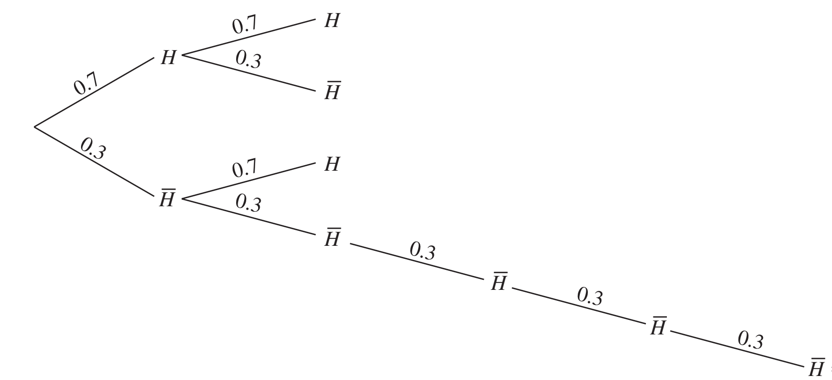
Example

A student is practising her goal scoring for soccer. The probability that the ball hits the net on any particular attempt is 0.7 and she does not improve with practice.

a Find how many balls should be kicked so that the probability that she hits the net at least once is greater than 0.995.

b Find how many balls should be kicked so that the probability that she does not hit the net is less than 0.001.

We begin by drawing part of the tree diagram. Because we do not know how many times she kicks the ball, the diagram potentially has an infinite number of branches.



a From the tree diagram we can see that the only branch of the tree where she never hits the net is the one marked *.

Thus

$$\begin{aligned}1-(0.3)^n &> 0.995\\ \Rightarrow 0.3^n &< 0.005\\ \Rightarrow \log 0.3^n &< \log 0.005\\ \Rightarrow n\log 0.3 &< \log 0.005\\ \Rightarrow n &> \frac{\log 0.005}{\log 0.3}\\ \Rightarrow n &> 4.40\ldots\\ \Rightarrow n &= 5 \text{ since } n\in\mathbb{N}\end{aligned}$$

Taking logs of both sides

Using the laws of logs from Chapter 5

The inequality changes because log 0.3 is negative.

b In this case the branch of the tree diagram that we are interested in is again the one marked *.

We want

$$\begin{aligned}0.3^n &< 0.001\\ \Rightarrow \log 0.3^n &< \log 0.001\\ \Rightarrow n\log 0.3 &< \log 0.001\\ \Rightarrow n &> \frac{\log 0.001}{\log 0.3}\\ \Rightarrow n &> 5.73\ldots\\ \Rightarrow n &= 6\end{aligned}$$

We will now look at two examples that bring together a number of these results.

Example

The results of a traffic survey on cars are shown below.

	Less than 3 years old	Between 3 and 6 years old	More than 6 years old
Grey	30	45	20
Black	40	37	17
White	50	30	31

a What is the probability that a car is less than 3 years old?
b What is the probability that a car is grey or black?
c Are these independent events?
d Given that a car is grey, what is the probability that it is less than 3 years old?
e Given that a car is more than 6 years old, what is the probability that it is white?

a Since there are 300 cars in the survey, the probability that a car is less than 3 years old is

$$\frac{30 + 40 + 50}{300} = \frac{2}{5}$$

b Since these are mutually exclusive events, the probability that it is grey or black is the probability that it is grey + the probability that it is black

$$= \frac{30 + 45 + 20}{300} + \frac{40 + 37 + 17}{300} = \frac{189}{300} = \frac{63}{100}$$

c We define the probability that a car is grey as P(G), the probability that a car is black as P(B) and the probability that it is less than 3 years old as P(X). If these events are independent then $P[(B \cup G)|X] = P(B \cup G)$.

$$P[(B \cup G)|X] = \frac{P[(B \cup G) \cap X]}{P(X)}$$

$$= \frac{\frac{70}{30}}{\frac{120}{300}} = \frac{7}{12}$$

Since this is not the same as $\frac{63}{100}$ the events are not independent.

d We require $P(X|G) = \dfrac{P(X \cap G)}{P(G)}$

$$= \frac{\frac{30}{300}}{\frac{95}{300}} = \frac{30}{95} = \frac{6}{19}$$

e We begin by defining the probability that a car is white as P(W) and the probability that it is more than 6 years old as P(Y).

We require $P(W|Y) = \dfrac{P(W \cap Y)}{P(Y)}$

$$= \frac{\frac{31}{300}}{\frac{68}{300}} = \frac{31}{68}$$

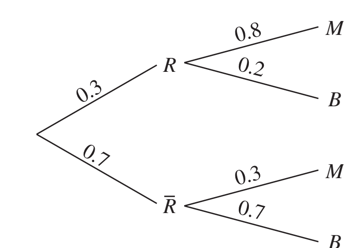
Example

Mike and Belinda are both keen cyclists and want to see who is the best cyclist. They do this by competing in a series of independent races, and decide that the best cyclist will be the one to win three races. These races only have the two of them as contestants. However, the probability of either of them winning a race is dependent on the weather. In the rain the probability that Mike will win is 0.8, but when it is dry the probability that Mike will win is 0.3. In every race the weather is either defined as rainy or dry. The probability that on the day of a race the weather is rainy is 0.3.

a Find the probability that Mike wins the first race.
b Given that Mike wins the first race, what is the probability that the weather is rainy?
c Given that Mike wins the first race, what is the probability that Mike is the best cyclist?

Let the probability that Mike wins a race be P(M), the probability that Belinda wins a race be P(B), and the probability that it rains be P(R).

a We begin by drawing a tree diagram.

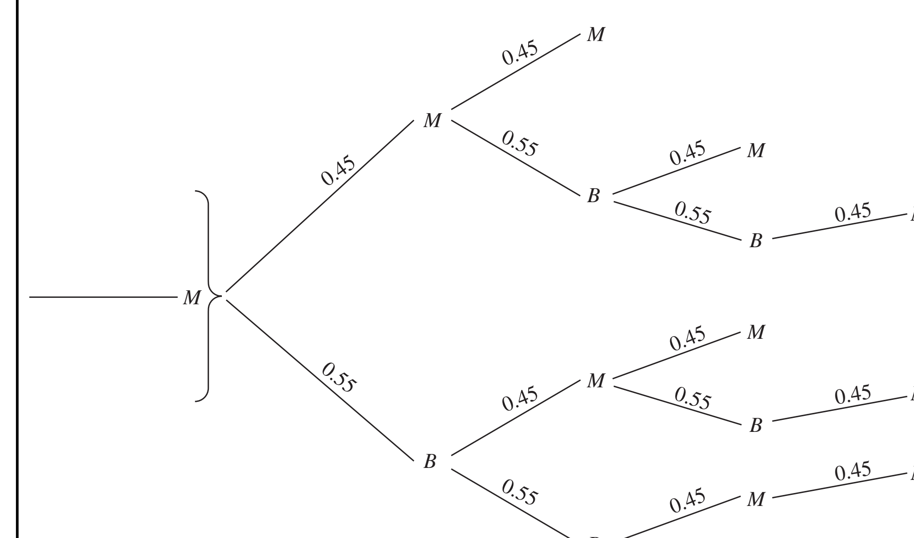


$P(M) = 0.3 \times 0.8 + 0.7 \times 0.3 = 0.45$

b We require $\dfrac{P(R \cap M)}{P(M)} = \dfrac{0.3 \times 0.8}{0.45} = \dfrac{8}{15}$

c We now draw a tree diagram showing the different ways that Mike and Belinda can win three races. We now know that $P(M) = 0.45$ and $P(B) = 0.55$ from part **a**.

Since it is given that Mike has won the first race we can ignore this part of the tree diagram.



Hence the probability that Mike is the best cyclist

$= (0.45 \times 0.45) + (0.45 \times 0.55 \times 0.45) + (0.45 \times 0.55 \times 0.55 \times 0.45) + (0.55 \times 0.45 \times 0.45) + (0.55 \times 0.45 \times 0.55 \times 0.45) + (0.55 \times 0.55 \times 0.45 \times 0.45)$

$= 0.609$

Exercise 4

1 Shonil is practising playing darts. The probability that he hits the dartboard is 0.6.

a Find the probability that he hits the dartboard at least once in four throws.

b How many darts must he throw in order that the probability that he hits the dartboard at least once is greater that 0.99?

c How many darts must Shonil throw in order that the probability that the dartboard is not hit is less than 0.01?

2 A die is biased such that the probability of throwing a six is 0.2.

a Find the probability of throwing at least one six in the first eight throws.

b How many times must the die be thrown in order that the probability of getting at least one six is greater than 0.995?

3 To proceed to the second round in a mathematics competition, Katie must get at least one of the four questions in the first round completely correct. The probability that she gets a question completely correct is 0.65. What is the probability that she proceeds to the second round?

4 Terri and Robyn are playing a game with a tetrahedral die. Terri goes first and they take turns at throwing the die. The first person to throw a one is the winner. What is the probability that Robyn wins?

5 Ayesha cycles to school. She has a choice of two routes, route A and route B. She is three times more likely to travel by route A than route B. If she travels by route A the probability that she will be late is 0.1 and if she travels by route B the probability that she will be late is 0.15. On a particular Monday Ayesha is late for school. Use Bayes' theorem to find the probability that she travelled by route A.

6 Ali should take two examinations, one in mathematics and one in English. On the day of the examinations, he takes only one. He is five times more likely to take mathematics than English. If he takes mathematics the probability that he passes is 0.9 and if he takes English the probability that he passes is 0.8. Ali passes the examination. Use Bayes' theorem to find the probability that he took mathematics.

7 Nicolle goes shopping to buy a present for her partner Ian. She has the choice of buying him a book or a DVD. She is four times more likely to buy him a book than a DVD. If she buys a book the probability that she pays with cash is 0.4 and if she buys a DVD the probability that she pays cash is 0.65. She pays cash for the present. Use Bayes' theorem to find the probability that Nicolle bought Ian a book.

8 In class 12A there are 30 students. 12 of the students hope to go to university A, 8 students hope to go to university B, 4 students hope to go to university C and 6 students hope to go to university D.

a Four students are picked at random.

i What is the probability that all four hope to go to university A?

ii What is the probability that all four hope to go to the same university?

iii Given that the first person picked hopes to go to university A, what is the probability that the other three hope to go to university B?

b Find the probability that exactly four students will be selected before a student who hopes to go to university C is selected.

9 Bill and David decide to go out for the day. They will either go to the beach or go to the mountains. The probability that they will go to the beach is 0.4. If they go to the beach the probability that they will forget the sunscreen is 0.1 and if they go to the mountains the probability they will forget the sunscreen is 0.35. They forget the sunscreen. Use Bayes' theorem to determine the probability that they go to the mountains.

10 Jerry and William play squash every week. In a certain week the probability that Jerry will win is 0.6. In subsequent weeks the probabilities change depending on the score the week before. For the winner, the probability of winning the following week increases by a factor of 1.05. For the loser, the probability of winning the following week remains unchanged.

a What is the probability that Jerry wins the first week and loses the second week?

b What is the probability that William wins for three consecutive weeks?

c Given that Jerry wins the first week, what is the probability that William wins for the following two weeks?

d How many games must William play to have less than a 1% chance of always winning?

11 In a class of students, there are five students with blue eyes, seven students with brown eyes, four students with hazel eyes and four students with green eyes. It is found in this class that it is only boys who have either blue or green eyes and only girls who have brown or hazel eyes. Three students are chosen at random.

a What is the probability that all three have blue eyes?

b What is the probability that exactly one student with brown eyes is chosen?

c What is the probability that two girls are chosen given that exactly one blue-eyed boy is chosen?

d What is the probability that the group contains exactly one hazel-eyed girl or exactly one green-eyed boy or both?

12 Arnie, Ben and Carl are going out for the night and decide to meet in town. However, they cannot remember where they decided to meet. Arnie cannot remember whether they were meeting in the square or outside the cinema. To make a decision he flips an unbiased coin. Ben cannot remember whether they were meeting outside the cinema or outside the theatre. He also flips a coin, but the coin is biased. The probability that it will land on a head is 0.6. If it lands on a head he will go to the cinema, but if it lands on a tail he will go to the theatre. Carl knows they are meeting either in the square, outside the cinema or outside the theatre. He flips an unbiased coin. If it lands on heads he goes to the theatre, but if it lands on tails he flips again. On the second flip, if it lands on heads he goes to the square and if it lands on tails he goes to the theatre.

a What is the probability that Arnie and Ben meet?

b What is the probability that Ben and Carl meet?

c What is the probability that all three meet?

d Given that Carl goes to the cinema, what is the probability that all three will meet?

13 To promote the sale of biscuits, a manufacturer puts cards showing pictures of celebrities in the packets. There are four different celebrities on the cards. Equal numbers of cards showing each celebrity are randomly distributed in the packets and each packet has one card.

a If Ellen buys three packets of biscuits, what is the probability that she gets a picture of the same celebrity in each packet?

b If she buys four packets and the first packet she opens has a card with a picture of celebrity A, what is the probability that the following three packets will contain cards with celebrity B on them?

c Ellen's favourite celebrity is celebrity B. How many packets must she buy to have at least a 99.5% chance of having at least one picture of celebrity B?

14 At the ninth hole on Sam's local golf course he has to tee off over a small lake. If he uses a three wood, the probability that the ball lands in the lake is 0.15. If he uses a five wood, the probability that the ball lands in the lake is 0.20. If he uses a three iron, the probability that the ball lands in the lake is 0.18. If he tees off and the shot lands in the lake, he has to tee off again.

a What is the probability that if he tees off with a three wood, he needs three shots to get over the lake?

b Sam decides that if his shot lands in the lake on the first tee off, on the next tee off he will use a different club. He uses the three wood for the first shot, the five wood for the second, the three iron for the third and then returns to the three wood for the fourth and continues in that order.

i What is the probability that he successfully tees off over the lake on his second use of the three wood?

ii Given that he uses a three wood twice, what is the probability that he successfully hits over the lake on his sixth shot?

15 An author is writing a new textbook. The probability that there will be a mistake on a page is $\frac{1}{20}$ if he is writing in the evening and $\frac{1}{30}$ if he is writing in the morning.

a What is the probability that if he is writing in the morning there is one mistake on each of three consecutive pages?

b How many pages must he write in the evening for there to be a greater than 99% chance of at least one error?

c He writes page 200 of the book in the morning, page 201 in the evening and page 202 in the morning. Given that page 201 has no mistakes on it, what is the probability that both pages 200 and 202 have a mistake on them?

20.5 Permutations and combinations

We met the idea of permutations and combinations in Chapter 6. A combination of a given number of articles is a set or group of articles selected from those given where the order of the articles in the set or group **is not** taken into account. A permutation of a given number of articles is a set or group of articles selected from those given where the order of the articles in the set or group **is** taken into account. At that point we looked at straightforward questions and often used the formulae to calculate the number of permutations or combinations. We will now look at some more complicated examples where the formulae do not work directly.

Permutations

Example

How many arrangements can be made of three letters chosen from the word PLANTER if the first letter is a vowel and each arrangement contains three different letters?

We split this into two separate calculations.
Assume we begin with the letter A. Hence the other two letters can be chosen in 6×5 ways $= 30$ ways.
If we begin with the letter E, then the other two letters can also be chosen in 30 ways.
Since these are the only two possibilities for beginning with a vowel, there are $30 + 30 = 60$ possible permutations.

Example

How many three-digit numbers can be made from the set of integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ if

a the three digits are all different
b the three digits are all the same
c the number is greater than 600
d the number is even and each digit can only be used once?

a The first digit can be chosen in nine ways.
The second digit can be chosen in eight ways.
The third digit can be chosen in seven ways.
$\Rightarrow$ Total number of three-digit numbers that are all different
$= 9 \times 8 \times 7 = 504$

b If the three digits are all the same then there are nine possible three-digit numbers, since the only possibilities are 111, 222, 333, 444, 555, 666, 777, 888 and 999.

c If the number is greater than 600 then there are only four choices for the first digit: 6, 7, 8 or 9.
The second and third digits can each be chosen in nine ways.
$\Rightarrow$ Total number of three-digit numbers $= 4 \times 9 \times 9 = 324$

d In this case we start with the last digit as this is the one with the restriction.
The last digit can be chosen in four ways.
The other two digits can be chosen in eight ways and seven ways respectively.
$\Rightarrow$ Total number of three-digit numbers $= 4 \times 8 \times 7 = 224$

Example

Find the number of two- and three-digit numbers greater than 20 that can be made from 1, 2 and 3, assuming each digit is only picked once.

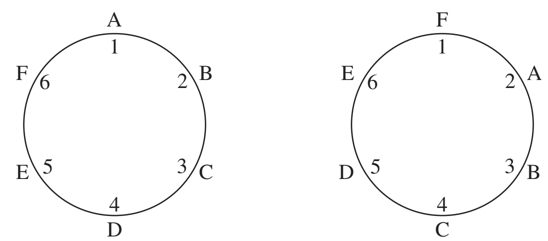
We split this into the two separate problems of finding the two-digit numbers and the three-digit numbers.
For the two-digit numbers, the first digit must be either a 2 or a 3. Hence there are two ways of picking the first digit. The second digit can also be picked in two ways as we can choose either of the two left.
$\Rightarrow$ Total number of two-digit numbers $= 2 \times 2 = 4$
For the three-digit number, this is just the number of permutations, which is $3! = 6$.
$\Rightarrow$ Total number of two- and three-digit numbers greater than $20 = 4 + 6 = 10$

Example

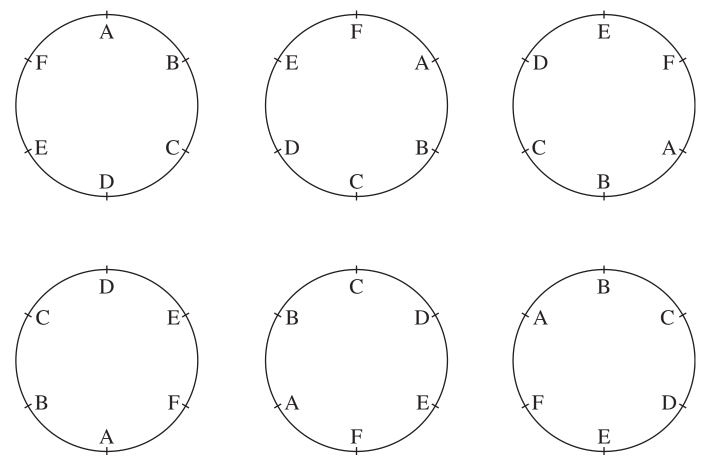
In how many ways can six people be sat around a circular dining table?

At first this appears to be a simple permutation, where the answer is 6! However, if we look at the two situations below, where the chairs are labelled from 1 to 6 and the people from A to F, we can see that they appear as different permutations, but are actually the same.

Because of the actual situation, every person has the same people on either side in both cases. If the chairs had been distinguishable, then this would no longer be the case.



Hence for every permutation of people sat around the table, there are five more permutations which are the same and hence the answer is six times too big. These are shown in the diagram below.



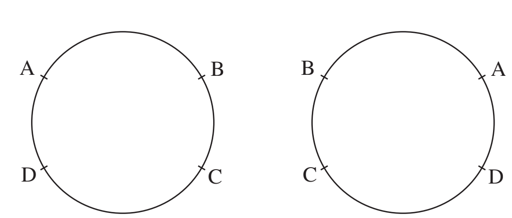
Therefore the number of ways that six people can be sat around a circular dining table is $\frac{6!}{6} = 5! = 120$.

For any situation like this the answer can be generalised to $(n - 1)!$

Example

Jenny is making a necklace. In how many ways can 4 beads chosen from 12 beads be threaded on a string?

This is similar to the example above. As with the example above the answer $12 \times 11 \times 10 \times 9$ needs to be divided by 4 because of the repetitions caused by the fact that it is on a circle. However in this situation there is another constraint because the necklace can be turned over giving an equivalent answer. In the diagrams below, these two situations are actually the same, but appear as two separate permutations of the answer.



Hence we need to divide the answer by 2.

Therefore the number of permutations is $\frac{12 \times 11 \times 10 \times 9}{4 \times 2} = 1485$

Example

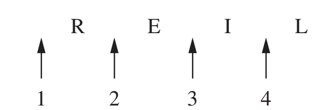
a Find the number of arrangements of the letters of the word LITTER.
b Find the number of arrangements where the T's are together.
c Find the number of arrangements where the T's are separated.

a In this question we treat it as a simple permutation and hence the answer would appear to be 6! However, the two T's are indistinguishable and hence LIT_1T_2LE and LIT_2T_1LE are actually the same arrangement but appear as two separate permutations. As this happens in every single case the number of arrangements is $\frac{6!}{2!} = 360$.

b With the T's together we treat the two T's as one letter. If we give TT the symbol Θ, then we are finding the permutations of LI Θ ER, which are $5! = 120$ arrangements.

c For the T's separated, we remove the T's initially and find the number of permutations of LIER which is 4!

If we now consider the specific permutation REIL, then the two T's can be placed in two of five positions. This is shown in the diagram below.



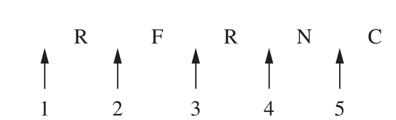
Hence for the permutation REIL there are ${}^5C_2 = 10$ ways of positioning the T's. As this can happen with each of the 4! permutations of the four letters, then the total number of permutations is $4! \times 10 = 240$ arrangements.
We could also think about this another way. As we know the total number of arrangements is 360 and the T's either have to be together or separated then the number of arrangements where they are separated is 360 minus the number of arrangements where they are together.
This gives $360 - 120 = 240$ arrangements.

Example

Find the number of arrangements of the letters in the word REFERENCE where the E's are separated.

We begin by considering RFRNC. The number of arrangements of these letters is $\frac{5!}{2!}$ since the two R's are indistinguishable.

Again considering one possible arrangement of the letters, say RFNRC, the positions that the three E's can take are shown below.



We need to find the number of combinations of four from six positions, which is ${}^6C_4 = 15$.

Hence the number of arrangements where the E's are separated is

$\frac{5!}{2!} \times 15 = 900$.

Combinations

Example

A team of 4 children is to be selected from a class of 20 children, to compete in a quiz game. In how many ways can the team be chosen if

a any four can be chosen

b the four chosen must include the oldest in the class?

a This is a straightforward combination with an answer of ${}^{20}C_4 = 4845$.

b In this situation we remove the oldest in the class since this child has to be part of every group. Hence the problem is actually to find how many teams of 3 children can be found from 19. Therefore the number of teams is ${}^{19}C_3 = 969$.

Example

Ten students in a class are divided into two groups of five to play in a five-a-side soccer tournament. In how many ways can the two teams of five be selected?

This appears to be very similar to the example above, but there is a subtle difference. The number of ways of selecting a team of five is ${}^{10}C_5 = 252$. Let us imagine that the chosen team is ABCDE. Hence the other team would automatically be FGHIJ. However another possible combination of a team of five would be FGHIJ and this would then automatically select the other team as ABCDE. In other words the calculation picks each pair of teams twice. Hence the actual number of teams is $\frac{{}^{10}C_5}{2} = 126$.

Example

Anisa goes into her local supermarket and finds that there are 20 different types of chocolate on offer and 15 different types of soft drink. She wants to buy seven different bars of chocolate and four different cans of soft drink for herself and her friends. Find the number of different ways in which she can do this.

The number of ways she can choose seven bars of chocolate is ${}^{20}C_7 = 77\,520$.
The number of ways she can choose four cans of soft drink is ${}^{15}C_4 = 1365$.
Since with any particular combination of chocolate bars she can put all the particular combinations of cans of soft drink, the total number of choices is $77\,520 \times 1365 = 105\,814\,800$.

Example

A box contains four red, two blue, one yellow and one pink ball. How many different selections of three balls may be made?

All three the same: The only possibility here is three red balls and hence there is only one way of doing this.
Two the same, one different: There are two possibilities for two the same, red and blue. The third ball can then be chosen from any of the others, so there are three possibilities because we cannot choose the same colour again. Therefore the total number of ways is $2 \times 3 = 6$.
All three different: Since there are four different colours of ball this is ${}^{4}C_3 = 4$.
Hence the number of different selections that can be made is $1 + 6 + 4 = 11$.

Unlike the previous example, we added the combinations here as opposed to multiplying them.

Exercise 5

1 In how many ways can six different files be arranged in a row on a desk?
2 In how many ways can two boys and two girls be chosen from a group of 15 boys and 18 girls?
3 In how many ways can four different letters be put in four different envelopes?
4 In how many ways can three different coats be arranged on five hooks in a row?
5 Giulia has ten different mathematics books and four different chemistry books. In how many ways can she arrange seven of the mathematics books and one chemistry book on a shelf if the chemistry book must always be at one end?
6 In how many ways can the letters of the word PHOTOGRAPH be arranged?
7 Two sets of books contain seven different novels and four different autobiographies. In how many ways can the books be arranged on a shelf if the novels and the autobiographies are not mixed up?
8 I now many ways can ten different examinations be arranged so that the two mathematics examinations are not consecutive and the two French examinations are not consecutive?
9 Given that each digit can be used more than once, how many two-digit numbers can be made from the set $\{2, 4, 6, 7, 8, 9\}$ if
  a any two digits can be used
  b the two digits must be the same
  c the number must be odd
  d the number must be greater than 60?
10 A quiz team of five students is to be chosen from nine students. The two oldest students cannot both be chosen. In how many ways can the quiz team be chosen?
11 Consider the letters of the word DIFFICULT.
  a How many different arrangements of the letters can be found?
  b How many of these arrangements have the two I's together and the two F's together?
  c How many of the arrangements begin and end with the letter F?
12 Margaret wants to put eight new plants in her garden. They are all different.
  a She first of all decides to plant them in a row. In how many ways can she do this?
  b She then decides that they would look better in a circle. In how many ways can she do this?
  c She now realizes that two of the plants are identical. How many arrangements are there for planting them in a row and for planting them in a circle?
13 **a** How many different arrangements of the word ARRANGEMENT can be made?
  b How many arrangements are there which start with a consonant and end with a vowel?

14 Jim is having a dinner party for four couples.
a In how many ways can the eight people be seated at Jim's circular dining table?
b John and Robin are a couple, but do not want to sit next to each other at the dinner party. In how many ways can the eight people now be seated?
c Jim decides that the two oldest guests should sit next to each other. In how many ways can the eight people now be seated?

15 **a** How many numbers greater than 300 can be made from the set $\{1, 2, 5, 7\}$ if each integer can be used only once?
b How many of these numbers are even?

16 **a** A local telephone number has seven digits and cannot start with zero. How many local numbers are there?
b The telephone company realizes that they do not have enough numbers. It decides to add an eighth digit to each number, but insists that all the eight-digit numbers start with an odd number and end with an even number. The number still cannot start with a zero. Does this increase or decrease the number of possible telephone numbers and if so by how many?

17 **a** How many different arrangements are there of the letters of the word INQUISITION?
b How many arrangements are there where the four I's are separate?
c How many arrangements are there where the S and the T are together?

18 Five different letters are written and five different envelopes are addressed. In how many ways can at least one letter be placed in the wrong envelope?

19 On an examination paper of 20 questions a student obtained either 6 or 7 marks for each question. If his total mark is 126, in how many different ways could he have obtained this total?

20 Four boxes each contain six identical coloured counters. In the first box the counters are red, in the second box the counters are orange, in the third box the counters are green and in the fourth box the counters are purple. In how many ways can four counters be arranged in a row if
a they are all the same
b three are the same and one is different
c they are all different
d there is no restriction on the colours of the counters?

21 **a** In how many ways can six different coloured beads be arranged on a ring?
b If two beads are the same colour, how many ways are there now?

22 **a** How many different combinations of six numbers can be chosen from the digits 1, 2, 3, 4, 5, 6, 7, 8 if each digit is only chosen once?
b In how many ways can the digits be divided into a group of six digits and a group of two digits?
c In how many ways can the digits be divided into two groups of four digits?

23 A shop stocks ten different types of shampoo. In how many ways can a shopper buy three types of shampoo if
a each bottle is a different type
b two bottles are the same type and the third is different?

24 A mixed team of 10 players is chosen from a class of 25 students. 15 students are boys and 10 students are girls. In how many ways can this be done if the team has five boys and five girls?

25 Find the number of ways in which ten people playing five-a-side football can be divided into two teams of five if Alex and Bjorn must be in different teams.

26 A tennis team of four is chosen from seven married couples to represent a club at a match. If the team must consist of two men and two women and a husband and wife cannot both be in the team, in how many ways can the team be formed?

27 Nick goes to the shop to buy seven different packets of snacks and four bottles of drink. At the shop he find he has to choose from 15 different packets of snacks and 12 different bottles of drink. In how many different ways can he make his selection?

28 In how many ways can three letters from the word BOOKS be arranged in a row if at least one of the letters is O?

20.6 Probability involving permutations and combinations

Sometimes we can use the idea of permutations and combinations when solving questions about probability.

Example

From a group of 12 people, 8 are chosen to serve on a committee.
a In how many different ways can the committee be chosen?
b One of the 12 people is called Sameer. What is the probability that he will be on the committee?
c Among the 12 people there is one married couple. Find the probability that both partners will be chosen.
d Find the probability that the three oldest people will be chosen.

a This is the combination ${}^{12}C_8 = 495$, which acts as the total possibility space.
b Since Sameer must be on the committee, we need to choose 7 people from 11. This can be done in ${}^{11}C_7 = 330$ ways. Hence the probability that Sameer will be on the committee is $\frac{330}{495} = \frac{2}{3}$.
c As in part **b**, we know that the married couple will be on the committee and need to choose six people from ten. This can be done in ${}^{10}C_6 = 210$ ways. Hence the probability that the married couple will be on the committee is $\frac{210}{495} = \frac{14}{33}$.
d Since the three oldest people have been chosen, we now choose the other five people from the nine remaining. This can be done in ${}^{9}C_5 = 126$ ways. Hence the probability that the three oldest people will be on the committee is $\frac{126}{495} = \frac{14}{55}$.

Example

Four letters are picked from the word EXAMPLES.
a How many different arrangements are there of the four letters?
b What is the probability that the arrangement of four letters will not contain a letter E?
c What is the probability that the arrangement will contain both of the letter E's?

d Given that the arrangement contains both of the letter E's, what is the probability that the two letter E's will be separated?

a Because of the repetition of the letter E we need to do this in groups.

Consider XAMPLS. The number of arrangements of four letters is ${}^6P_4 = 360$.

Consider XAMPLSE with the condition that the arrangement must contain one E. To do this we find the number of permutations of three letters from XAMPLS $= {}^6P_3 = 120$. Within each of these permutations there are four positions that the E can take. This is shown below, using XAM as an example.

X A M

1 2 3 4

Hence the number of permutations that contain one E is $4 \times 120 = 480$.

Consider XAMPLSEE with the condition that the arrangement must contain two E's. To do this we find the number of permutations of two letters from XAMPLS $= {}^6P_2 = 30$. Within each of these permutations there are five positions that the two E's can take. These are shown below, using XA as an example.

EEXA EXEA EXAE

XAEE XEEA

The number of permutations that contain two E's is $5 \times 30 = 150$. Hence the total possible number of arrangements is $240 + 480 + 150 = 870$.

b Since the total number of arrangements is 870 and the number that do not contain an E is 360, the probability is $\frac{360}{870} = \frac{12}{29}$.

c Since the total number of arrangements is 870 and the number that contain both E's is 150, the probability is $\frac{150}{870} = \frac{5}{29}$.

d Because this is a conditional probability question we are limiting the sample space to only containing two E's. Consider the permutation PL. The two E's can be positioned as shown below.

EEPL EPEL EPLE

PLEE PEEL

Hence for any permutation of two letters, two out of the 5 cases will have the letter E's separated. This is true for every permutation of two letters. Hence the probability is $\frac{2}{5}$.

Exercise 6

1 A group of three boys and one girl is chosen from six boys and five girls.

a How many different groups can be formed?

b What is the probability that the group contains the oldest boy?

c What is the probability that it contains the youngest girl?

d Within the six boys there are two brothers. What is the probability that the group contains both brothers?

2 a In how many ways can ten people be sat around a circular table?

b Within the ten people there are two sisters.

i What is the probability that the sisters will sit together?

ii What is the probability that the sisters will not sit together?

c The ten people consist of five men and five women. What is the probability that no two men and no two women are sat next to each other?

3 Allen is predicting the results of six soccer matches.

a In how many ways can he predict exactly four correct results?

b His favourite team is A and they are playing in one of the six matches. Given that he predicts exactly four correct results, what is the probability that he will predict correctly the result of the match in which team A are playing?

4 Consider the letters of the word EATING.

a How many different arrangements of four letters can be formed?

b What is the probability that the four-letter arrangement contains the letter A?

c What is the probability that the four-letter arrangement contains either the letter T or the letter G, but not both?

5 At a local squash club there are 40 members. League A consists of six people.

a If league A is made up randomly from the 40 members, in how many different ways can league A be made?

b What is the probability that league A will contain the oldest member of the club?

c Given that league A contains the oldest member of the club, what is the probability that it also contains the youngest?

6 Six letters are picked from the word CULTURES.

a How many different arrangements of six letters can be formed?

b What is the probability that an arrangement contains exactly one U?

c Given that the arrangement contains both U's, what is the probability that both U's are together at the start of the arrangement?

7 a How many even numbers less than 500 can be formed from the digits 2, 4, 5, 7 and 9?

b An even number is picked at random.

i What is the probability that it is a two-digit number?

ii What is the probability that it is greater than 500?

iii What is the probability that it is a three-digit number beginning with 4?

8 Laura has ten plants to put in a row along the fence of her garden. There are four identical roses, four identical clematis and two identical honeysuckle.

a In how many different ways can she plant them along the fence?

b What is the probability that the four roses are all together?

c What is the probability that there is a honeysuckle at each end of the row?

d What is the probability that no clematis is next to another clematis?

9 A committee meeting takes place around a rectangular table. There are six members of the committee and six chairs. Each position at the table has different papers at that position.

a How many different arrangements are there of the six committee members sitting at the table?

b Two friends Nikita and Fatima are part of the committee. What is the probability that either of them sit at the table with papers A and B in front of them?

c The only two men on the committee are Steve and Martin. What is the probability that they sit in positions C and F respectively?

10 Jim is trying to arrange his DVD collection on the shelf. He has ten DVDs, three titles starting with the letter A, four titles starting with the letter C and three titles starting with the letter S. Even though they start with the same letter the DVDs are distinguishable from each other.

a In how many ways can Jim arrange the DVDs on the shelf?

b What is the probability that the four starting with the letter C will be together?

c What is the probability that no DVD starting with the letter A will be next to another DVD starting with the letter A?

d Given that the three DVDs starting with the letter S are together, what is the probability that the three DVDs starting with the letter A will be together?

Review exercise

A calculator may be used in all questions in this exercise where necessary.

1 In a group of 30 boys, they all have black hair or brown eyes or both. 20 of the boys have black hair and 15 have brown eyes. A boy is chosen at random.
a What is the probability that he has black hair and brown eyes?
b Are these two events independent, mutually exclusive or exhaustive? Give reasons.

2 Roy drives the same route to school every day. Every morning he goes through one set of traffic lights. The probability that he has to stop at the traffic lights is 0.35.
a In a five-day week, what is the probability that he will have to stop on at least one day?
b How many times does Roy have to go through the traffic lights to be able to say that there is a 95% chance that he will have to stop?

3 There are ten seats in a waiting room. There are six people in the room.
a In how many different ways can they be seated?
b In the group of six people, there are three sisters who must sit next to each other. In how many different ways can the group be seated? [IB May 06 P1 Q19]

4 Tushar and Ali play a game in which they take turns to throw an unbiased cubical die. The first one to throw a one is the winner. Tushar throws first.
a What is the probability that Tushar wins on his first throw?
b What is the probability that Ali wins on his third throw?
c What is the probability that Tushar wins on his *n*th throw?

5 A bag contains numbers. It is twice as likely that an even number will be drawn than an odd number. If an odd number is drawn, the probability that Chris wins a prize is $\frac{3}{16}$. If an even number is drawn, the probability that Chris wins a prize is $\frac{5}{16}$. Chris wins a prize. Use Bayes' theorem to find the probability that Chris drew an even number.

6 In how many ways can six different coins be divided between two students so that each student receives at least one coin? [IB Nov 00 P1 Q19]

7 **a** In how many ways can the letters of the word PHOTOGRAPH be arranged?
b In how many of these arrangements are the two O's together?
c In how many of these arrangements are the O's separated?

8 The local football league consists of ten teams. Team A has a 40% chance of winning any game against a higher-ranked team, and a 75% chance of winning any game against a lower-ranked team. If A is currently in fourth position, find the probability that A wins its next game. [IB Nov 99 P1 Q13]

9 Kunal wants to invite some or all of his four closest friends for dinner.
a In how many different ways can Kunal invite one or more of his friends to dinner?
b Mujtaba is his oldest friend. What is the probability that he will be invited?
c Two of his closest friends are Anna and Meera. What is the probability that they will both be invited?

10 The probability of Prateek gaining a grade 7 in Mathematics HL given that he revises is 0.92. The probability of him gaining a grade 7 in Mathematics HL given that he does not revise is 0.78. The probability of Prateek revising is 0.87. Prateek gains a grade 7 in Mathematics HL. Use Bayes' theorem to find the probability that he revised.

11 A committee of five people is to be chosen from five married couples. In how many ways can the committee be chosen if
a there are no restrictions on who can be on the committee
b the committee must contain at least one man and at least one woman
c the committee must contain the youngest man
d both husband and wife cannot be on the committee?

12 In a bilingual school there is a class of 21 pupils. In this class, 15 of the pupils speak Spanish as their first language and 12 of these 15 pupils are Argentine. The other 6 pupils in the class speak English as their first language and 3 of these 6 pupils are Argentine. A pupil is selected at random from the class and is found to be Argentine. Find the probability that the pupil speaks Spanish as his/her first language. [IB May 99 P1 Q8]

13 How many different arrangements, each consisting of five different digits, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, if
a each arrangement begins and ends with an even digit
b in each arrangement odd and even digits alternate? [IB Nov 96 P1 Q12]

14 Three suppliers A, B and C produce respectively 45%, 30% and 25% of the total number of a certain component that is required by a car manufacturer. The percentages of faulty components in each supplier's output are, again respectively, 4%, 5% and 6%. What is the probability that a component selected at random is faulty? [IB May 96 P1 Q4]

15 **a** How many different arrangements of the letters of the word DISASTER are there?
b What is the probability that if one arrangement is picked at random, the two S's are together?
c What is the probability that if one arrangement is picked at random, it will start with the letter D and finish with the letter T?

16 Note: In this question all answers must be given exactly as rational numbers.
a A man can invest in at most one of two companies, A and B. The probability that he invests in A is $\frac{3}{7}$ and the probability that he invests in B is $\frac{2}{7}$, otherwise he makes no investment. The probability that an investment yields a dividend is $\frac{1}{2}$ for company A and $\frac{2}{3}$ for company B. The performances of the two companies are totally unrelated. Draw a probability tree to illustrate the various outcomes and their probabilities. What is the probability that the investor receives a dividend and, given that he does, what is the probability that it was from his investment in company A?
b Suppose that a woman must decide whether or not to invest in each company. The decisions she makes for each company are independent and the probability of her investing in company A is $\frac{3}{10}$ while the probability of her investing in company B is $\frac{6}{10}$. Assume that there are the same probabilities of the investments yielding a dividend as in part **a**.
i Draw a probability tree to illustrate the investment choices and whether or not a dividend is received. Include the probabilities for the various outcomes on your tree.
ii If she decides to invest in both companies, what is the probability that she receives a dividend from at least one of her investments?
iii What is the probability that she decides not to invest in either company?
iv If she does not receive a dividend at all, what is the probability that she made no investment? [IB May 96 P2 Q4]

17 An advanced mathematics class consists of six girls and four boys.

a How many different committees of five students can be chosen from this class?

b How many such committees can be chosen if class members Jack and Jill cannot both be on the committee?

c How many such committees can be chosen if there must be more girls than boys on the committee? [IB Nov 95 P1 Q12]

18 Each odd number from 1 to $3n$, where $n \in \mathbb{N}$ and n is odd, is written on a disc and the discs are placed in a box.

a How many discs are there in the box?

b What is the probability, in terms of n, that a disc drawn at random from the box has a number that is divisible by 3? [IB May 95 P1 Q19]

19 A box contains 20 red balls and 10 white balls. Three balls are taken from the box without replacement. Find the probability of obtaining three white balls. Let p_k be the probability that k white balls are obtained. Show by evaluating p_0, p_1, p_2 and p_3 that $\sum_{k=0}^{k=3} p_k = 1$. [IB Nov 87 P1 Q19]

20 Pat is playing computer games. The probability that he succeeds at level 1 is 0.7. If he succeeds at level 1 the next time he plays he goes to level 2 and the probability of him succeeding is $\frac{2}{3}$ the probability of him succeeding on level 1. If he does not succeed he stays on level 1. If he succeeds on level 2 he goes to level 3 where the probability of him succeeding is $\frac{2}{3}$ the probability of him succeeding on level 2. If he fails level 2 he goes back to level 1 with the initial probability of success. If he succeeds on level 3 he goes to level 4 where the probability of success is again $\frac{2}{3}$ the probability of him succeeding on level 3. If he fails on level 3 he goes back to level 2 and the probability of success is again the same as it was before. This continues in all games that Pat plays.

a What is the probability that after the third game he is on level 2?

b What is the probability that after the fourth game he is on level 1?

c What is the probability that after the third game he is on level 2?

d Given that he wins the first game, what is the probability that after the fourth game he is on level 3?

21 In Kenya, at a certain doctor's surgery, in one week the doctor is consulted by 90 people all of whom think they have malaria. 50 people test positive for the disease. However, the probability that the test is positive when the patient does not have malaria is 0.05 and the probability that the test is negative when the patient has malaria is 0.12.

a Find the probability that a patient who tested positive in the surgery has malaria.

b Given that a patient has malaria, what is the probability that the patient tested negative?

c Given that a patient does not have malaria, what is the probability that the patient tested positive?