Whole Course Review Exercise

1 The equation of a straight line l is given by $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + m\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ where m is a parameter. The point P lies on l where $m = 2$.

a Find the angle between OP and l, where O is the origin.

b Find the equation of a line perpendicular to l, and passing through the point $Q(0, 4, -3)$ in Cartesian form.

c The point R has coordinates $(-3, 1, 3)$. Find an equation of a plane π passing through P, Q and R in scalar product form.

d Is the line l contained in, parallel to, or does it intersect the plane? If it intersects, determine the coordinates of the point of intersection.

e The points S and T are on l and are given by $m = \mu$ and $m = 3\mu$ respectively. Show that there is no value of μ for which OS and OT are perpendicular.

2 In a yearly school soccer tournament, records are kept of the number of goals scored by each team. The tournament has been running for 20 years. The results for the last year are summarised in the table below.

Number of goals	6	7	8	9	10	11
Frequency	3	5	6	7	4	3

a State the modal value.

b Calculate the median.

c Calculate the mean.

d Calculate the standard deviation.

e Calculate an unbiased estimate for the standard deviation of the number of goals in the last 20 years.

f Two of the teams are chosen at random. What is the probability that the sum of their scores is 17?

g Given that the sum of the scores for the two teams is 17, what is the probability that the difference of their scores is 3?

h These scores are used to model the predicted scores for the following year. Write down the probability distribution for X where X is the number of goals scored.

i Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

3 a Prove, using the method of mathematical induction,

that $\left(\dfrac{3 - i}{1 - 2i}\right)^{4n} = (-4)^n,\ n \in \mathbb{Z}$.

b Hence evaluate $\left(\dfrac{3 - i}{1 - 2i}\right)^{16}$.

c Let $z = \dfrac{3 - i}{1 - 2i}$. If z is a root of a quadratic equation, calculate the second root.

d Calculate $\sqrt[4]{z}$ in the form $re^{i\theta}$ and draw these roots on an Argand diagram.

4 **a** Find the roots of the equation $32x^6 - 48x^4 + 18x^2 - 1 = \frac{\sqrt{3}}{2}$.

b Use de Moivre's theorem to find $\cos 6\theta$ as a function of $\sin\theta$.

c Solve the equation $\cos 6\theta = \sin\frac{4\pi}{3}$ in the range $0 \le \theta < \pi$.

d Hence express the roots of the equation in part **a** in the form $\sin\left(\frac{n\pi}{36}\right)$.

5 **a** Find the equation of the normal to the curve $\frac{x^2}{25a^2} - \frac{y^2}{9a^2} = 1$ which is parallel to the line $y = -x$ and passes through $x = 1$.

b If $y = 3a\tan\theta$ find x as a function of θ in its simplest form.

c Find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and hence find $\frac{dy}{dx}$ as a function of θ.

d Hence or otherwise show that $\csc\theta = \frac{3x}{5y}$.

6 The functions f and g are defined by $f(x) = 3x + 1$, $g(x) = \frac{2x}{x+1}$, $x \neq -1$.

a Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

b Evaluate $\int_2^3 [(f \circ g)(x) - (g \circ f)(x)]\,dx$.

c Find the values of x for which $(f \circ g)(x) \le (g \circ f)(x)$.

d Find the value of a for which $\int_{-\frac{3}{5}}^{a} [(f \circ g)(x) - (g \circ f)(x)]\,dx = 0$.

e Find the volume of the solid of revolution when the area bounded by the curve $(f \circ g)(x) - (g \circ f)(x)$, the lines $x = 2$ and $x = 5$ and the x-axis is rotated through 2π around the x-axis.

f Sketch the curve $y = |(f \circ g)(x) - (g \circ f)(x)|$.

7 Consider the curve $y = \frac{2x - 1}{x + 1}$.

a Show algebraically that the curve has no maximum or minimum values.

b State the equations of any asymptotes.

c Hence sketch the curve.

The curve $g(x)$ has the following properties.

x	$g(x)$	$g'(x)$
1	5	3
2	−2	2

d **i** Find the derivative of $f(x)g(x)$ when $x = 1$.

ii Find the derivative of $(f \circ g)(x) - (g \circ f)(x)$ when $x = 2$.

8 The variables x, y and z satisfy the simultaneous equations

$$x - 3y + 4z = 4$$
$$2x - 4y + 2z = 7$$
$$3x - 5y = c$$

where c is a constant.

a Show that these equations do not have a unique solution.

b Find the value of c which makes these equations consistent.

c For this value of c, find the general solution to these equations.

d State the vector equation of the line of intersection, l, of the three planes in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

e Find the coordinates of the point of intersection of the line, l, and the plane $\mathbf{r}\cdot\begin{pmatrix}3\\4\\-2\end{pmatrix} = 6$.

f Find the angle between the line of intersection, l, and the plane $\mathbf{r}\cdot\begin{pmatrix}3\\4\\-2\end{pmatrix} = 6$.

9 **a** If $y = 2^{2x+1}$ show that $\frac{dy}{dx} = 2^{2x+2}\ln 2$.

b Find the equations of the tangent and the normal to the curve $2^{2x+1} = 2^{y-1} - 4$ at the point where $x = 0$.

c Find the solution to the simultaneous equations

$$2^{2x+1} = 2^{y-1} - 4$$
$$y = 2x$$

giving the answer in the form $p - r\log_2 q$ where p, q, and $r \in \mathbb{Z}$.

10 The weight of the contents of packets of flour are normally distributed with a mean of 500 grams and a standard deviation of 50 grams.

a Find the probability that a randomly chosen packet of flour has a weight greater than 530 grams.

b For a packet of flour to be accepted, its weight must lie between 490 grams and 515 grams. If a company produces 1500 packets per day, how many packets are rejected?

c The company decides that there must only be a 3% chance of the packet of flour weighing less than 490 grams and a 2% chance of it being more than 515 grams. Calculate the new mean and standard deviation.

d In this case, calculate the probability that the weight of a bag of flour is more than 500 grams.

e If five bags of flour are selected at random, what is the probability that:

i exactly two of them

ii more than three of them

have a weight of more than 500 grams?

11 **a** Sketch the graph of $f(x) = \arctan x$, illustrating asymptotes and intercepts.

b Use integration by parts to find the area between $f(x)$, the x axis and the lines $x = 0$ and $x = 1$.

c Sketch the graph of $y = |f(x)|$.

d Show that the area between the graph of $y = |f(x)|$, the x axis and the lines $x = -\sqrt{3}$ and $x = 1$ is equal to $\frac{4\sqrt{3}\pi + 3\pi - 6\ln 8}{12}$.

12 A function is defined by $f(x) = \frac{\ln x}{x^2}$ for $x > 0$.

a **i** Find $f'(x)$.

ii Find $f''(x)$.

b Find the exact value of x for which $f'(x) = 0$.

c Show that this value of x provides a local maximum.

d State the coordinates of this turning point.

Whole Course Review Exercise

13 **a** Solve $2x\frac{dy}{dx} = x^2y$ given that when $x = 0, y = 5$.

b Find the area given by $\int_{-1}^{0} x^2 \cos x - 2xe^{3x}\, dx$.

c Find the volume of revolution of the solid formed when this area is rotated by 2π about the x axis.

14 The velocity of a particle moving in a straight line is given by $v(t) = 4t - kt\sin(kt)$, where t is time in seconds, and v is velocity in $\mathrm{m\,s^{-1}}$.

a Show that $\frac{dv}{dt} = 4 - k^2t\cos(kt) - k\sin(kt)$.

b Let $k = 1$.

i What was the acceleration of the particle after $\frac{\pi}{2}$ seconds?

ii Show that the distance travelled in this time was $\frac{\pi}{2}(\pi - 1)$ metres.

iii Does the particle ever come to rest? Justify your answer.

15 **a** The first three terms of an arithmetic sequence are $x + 2, 3x + 2, 7x - 4$. Find the first term and the common difference of the sequence.

b **i** The third term of a geometric sequence is 10 and the sixth term is $\frac{5}{4}$. Find the first term and the common ratio of the sequence.

ii Hence find the sum to infinity of this sequence.

c Consider the geometric series

$\sin 2\theta + 2\sin 2\theta\cos^2\theta + 4\sin 2\theta\cos^4\theta + 8\sin 2\theta\cos^6\theta + \cdots$

Show that, for the set of values of θ for which the sum to infinity exists, the sum to infinity of the progression is $-\tan 2\theta$.

16 For the function given by $f(x) = e^{kx}(x - 2)$,

a **i** show that $f'(x) = e^{kx}(1 + k(x - 2))$

ii where $f^{(n)}(x)$ denotes the nth derivative, prove by mathematical induction that $f^{(n)}(x) = k^{n-1}e^{kx}(n + k(x - 2))\ \forall n \in \mathbb{Z}^+$.

b Let $k = 2$.

i Find the coordinates of the point where $f'(x) = 0$.

ii Find the point of inflexion.

c Let $k = 1$.

Show that the area between the curve, the x axis, and the lines $x = -2$ and $x = 2$ is $-e^2 + \frac{5}{e^2}$.

17 The function f is defined by $f(x) = \frac{x^2}{x + 3}$, $x \neq -3$.

a Obtain algebraically the asymptotes of the graph of f.

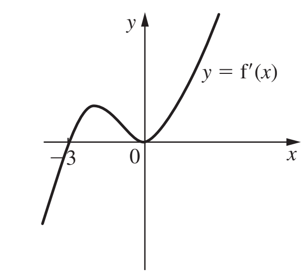
b Find the points where $f'(x) = 0$ and classify their nature.

c Sketch the curve showing clearly the features found in parts ***a*** and ***b***.

d Write down the coordinates of the stationary points of the graph of $g(x) = 10 + |f(x)|$.

Whole Course Review Exercise

18 For the graph below of $y = f'(x)$.



a **i** Sketch the graph of $y = f''(x)$.

ii Sketch a possible graph of $y = f(x)$.

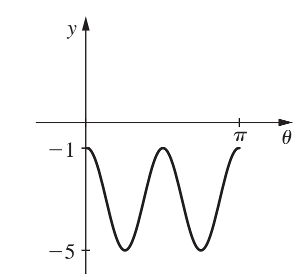
b For $k(x) = \frac{2x + 1}{x - 2}$, $p(x) = \frac{1}{2x}$,

i sketch the graph of $k(x) = \frac{2x + 1}{x - 2}$

ii find $k(p(x))$ in simplest form

iii hence give the largest possible domain for $k(p(x))$.

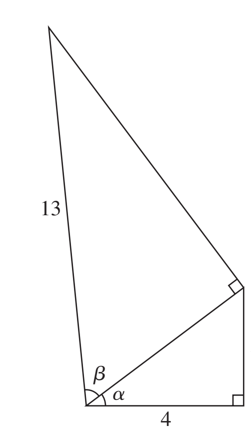
19



a What is the equation of this graph?

b Find the intersection of the above graph with the line $y = -4$ for $0 \leq \theta \leq \pi$.

c For the diagram below, find the exact value of $\sin(\alpha + \beta)$.



d Prove that $2\cos 2\theta - \cos 4\theta = 4\sin^4\theta + 1$.

20 **a** Solve $z^3 + 3z^2 + 50 = 0$, $z \in \mathbb{C}$.

b Express $z = \sqrt{3} - \mathrm{i}$ in $re^{\mathrm{i}\theta}$ form.

c Using de Moivre's theorem, find z^9, expressing your answer in $a + \mathrm{i}b$ form.

d Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

Using de Moivre's theorem, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

e By expanding $\left(z + \dfrac{1}{z}\right)^3$, show that $\cos^3\theta = \dfrac{1}{4}(\cos 3\theta + 3\cos\theta)$.