11 Matrices

The concept of matrices and determinants was probably first understood by the Babylonians, who were certainly studying systems of linear equations. However, it was *Nine Chapters on the Mathematical Art*, written during the Han Dynasty in China between 200 BC and 100 BC, which gave the first known example of matrix methods as set up in the problem below.

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. One of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

The author of the text sets up the coefficients of the system of three linear equations in three unknowns as a table on a "counting board" (see Matrix 1). The author now instructs the reader to multiply the middle column by 3 and subtract the right column *as many times as possible*. The right column is then subtracted *as many times as possible* from 3 times the first column (see Matrix 2). The left-most column is then multiplied by 5 and the middle column is subtracted *as many times as possible* (see Matrix 3).

This chapter will reveal that we now write linear equations as the rows of a matrix rather than columns, but the method is identical.

1	2	3
2	3	2
3	1	1
26	34	39

Matrix 1

0	0	3
4	5	2
8	1	1
39	24	39

Matrix 2

0	0	3
0	5	2
36	1	1
99	24	39

Matrix 3

Looking at the left-hand column, the solution can now be found for the third type of corn. We can now use the middle column and substitution to find the value for the second type of corn and finally the right-hand column to find the value for the first type of corn. This is basically the method of Gaussian elimination, which did not become well known until the early 19th century and is introduced in this chapter.

11.1 Introduction to matrices

Definitions

Elements: The numbers or symbols in a matrix.

Matrix: A rectangular array of numbers called entries or elements.

Row: A horizontal line of elements in the matrix.

Column: A vertical line of elements in the matrix.

Order: The size of the matrix. A matrix of order $m \times n$ has m rows and n columns.

Hence the matrix $A = \begin{pmatrix} 2 & 4 & -1 \\ 3 & 7 & 1 \end{pmatrix}$ has six elements, two rows, three columns, and its order is 2×3.

A matrix is usually denoted by a capital letter.

A square matrix is one that has the same number of rows as columns.

The most elementary form of matrix is simply a collection of data in tabular form like this:

	Week		
Sales of	**1**	**2**	**3**
Butter	75	70	82
Cheese	102	114	100
Milk	70	69	72

This data can be represented using the matrix $\begin{pmatrix} 75 & 70 & 82 \\ 102 & 114 & 100 \\ 70 & 69 & 72 \end{pmatrix}$.

Operations

Equality

Two matrices are equal if they are of the same order and their corresponding elements are equal.

Example

Find the value of a if $\begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} a & 5 \\ -3 & 4 \end{pmatrix}$.

Clearly in this case, $a = 2$.

Addition and subtraction

To add or subtract two or more matrices, they must be of the same order. We add or subtract corresponding elements.

Example

Evaluate $\begin{pmatrix} 2 & 4 & 3 \\ -1 & 3 & 7 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 7 \\ 4 & -4 & -2 \end{pmatrix}$.

In this case the answer is $\begin{pmatrix} 8 & 2 & 10 \\ 3 & -1 & 5 \end{pmatrix}$.

If the question appears on a calculator paper and does not involve variables, then a calculator can be used to do this.

We can now return to the example of a matrix given at the beginning of the chapter where the table

	Week		
Sales of	**1**	**2**	**3**
Butter	75	70	82
Cheese	102	114	100
Milk	70	69	72

can be represented as the matrix $\begin{pmatrix} 75 & 70 & 82 \\ 102 & 114 & 100 \\ 70 & 69 & 72 \end{pmatrix}$.

If this were to represent the sales in one shop, and the matrix $\begin{pmatrix} 79 & 78 & 79 \\ 97 & 101 & 109 \\ 81 & 75 & 74 \end{pmatrix}$ represents the sales in another branch of the shop, then adding the matrices together would give the total combined sales i.e. $\begin{pmatrix} 154 & 148 & 161 \\ 199 & 215 & 209 \\ 151 & 144 & 146 \end{pmatrix}$.

Multiplication by a scalar

The scalar outside the matrix multiplies every element of the matrix.

This can be done by calculator, but is probably easier to do mentally.

Example

Evaluate $-3\begin{pmatrix} 1 & 2 & -4 \\ 3 & 1 & 2 \\ -1 & 3 & 5 \end{pmatrix}$.

In this case the answer is $\begin{pmatrix} -3 & -6 & 12 \\ -9 & -3 & -6 \\ 3 & -9 & -15 \end{pmatrix}$.

Multiplication of matrices

To multiply two matrices there are a number of issues we need to consider. In matrix multiplication we multiply each row by each column, and hence the number of columns in the first matrix must equal the number of rows in the second matrix. Multiplying an $m \times n$ matrix by an $n \times p$ matrix is possible because the first matrix has n columns and the second matrix has n rows. If this is not the case, then the multiplication cannot be carried out. The answer matrix has the same number of rows as the first matrix and the same number of columns as the second.

$$(m \times n) \times (n \times p)$$

This is the size of the answer matrix, $m \times p$.

This tells us the matrix can be multipled.

To find the element in the first row and first column of the answer matrix we multiply the first row by the first column. The operation is the same for all other elements in the answer. For example, the answer in the second row, third column of the answer comes from multiplying the second row of the first matrix by the third column of the second matrix.

The matrices $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \\ t & u \end{pmatrix}$ are 2×3 and 3×2, so they can be multiplied to find AB, which will be a 2×2 matrix.

In this case $AB = \begin{pmatrix} ap + br + ct & aq + bs + cu \\ dp + er + ft & dq + es + fu \end{pmatrix}$. BA could also be found, and would be a 3×3 matrix.

If we consider the case of $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $D = \begin{pmatrix} t \\ u \\ v \end{pmatrix}$, then it is not possible to find either CD or DC since we have a 2×2 matrix multiplied by a 3×1 matrix or a 3×1 matrix multiplied by a 2×2 matrix.

Example

Evaluate $\begin{pmatrix} 3 & 6 \\ -2 & 4 \\ 1 & -5 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$.

We have a 4×2 matrix multiplied by a 2×2 matrix, and hence they can be multiplied and the answer will be a 4×2 matrix.

$$\begin{pmatrix} 3 & 6 \\ -2 & 4 \\ 1 & -5 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} (3 \times 5) + (6 \times 2) & (3 \times -1) + (6 \times 3) \\ (-2 \times 5) + (4 \times 2) & (-2 \times -1) + (4 \times 3) \\ (1 \times 5) + (-5 \times 2) & (1 \times -1) + (-5 \times 3) \\ (3 \times 5) + (-7 \times 2) & (3 \times -1) + (-7 \times 3) \end{pmatrix}$$

$$= \begin{pmatrix} 27 & 15 \\ -2 & 14 \\ -5 & -16 \\ 1 & -24 \end{pmatrix}$$

Example

If $A = \begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}$ find A^2.

As with algebra, A^2 means $A \times A$.

Hence $A^2 = \begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 13 & -8 \\ -32 & 29 \end{pmatrix}$

Example

Find the value of x and of y if $\begin{pmatrix} 3 & -1 \\ x & 2 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ y & -1 \end{pmatrix} = \begin{pmatrix} 3 & 19 \\ -4 & -32 \end{pmatrix}$.

Multiplying the left-hand side of the equation

$$\Rightarrow \begin{pmatrix} 6 - y & 19 \\ 2x + 2y & 6x - 2 \end{pmatrix} = \begin{pmatrix} 3 & 19 \\ -4 & -32 \end{pmatrix}$$

Equating elements:

$$\begin{aligned} 6 - y &= 3 \\ \Rightarrow y &= 3 \\ 2x + 2y &= -4 \\ \Rightarrow 2x + 6 &= -4 \\ \Rightarrow x &= -5 \end{aligned}$$

We now check $6x - 2 = -32$, which is true.

Example

If $A = \begin{pmatrix} 5 & 9 \\ -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & k \\ 0 & 2 \end{pmatrix}$, find $-2A(B - 2C)$.

We begin by finding $B - 2C = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 2k \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -5 - 2k \\ 1 & -6 \end{pmatrix}$

and $-2A = \begin{pmatrix} -10 & -18 \\ 4 & -2 \end{pmatrix}$

Hence $-2A(B - 2C) = \begin{pmatrix} -10 & -18 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 0 & -5 - 2k \\ 1 & -6 \end{pmatrix}$

$$= \begin{pmatrix} -18 & 158 + 20k \\ -2 & -8k - 8 \end{pmatrix}$$

Commutativity

Commutativity is when the result of an operation is independent of the order in which the elements are taken. Matrix multiplication is not commutative because in general $AB \neq BA$. In many cases multiplying two matrices is only possible one way or, if it possible both ways, the matrices are of different orders. Only in the case of a square matrix is it possible to multiply both ways and gain an answer of the same order, and even then the answers are often not the same.

Example

If $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ 3 & 4 \end{pmatrix}$, find:

a) AB
b) BA

a) $AB = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -8 \\ 21 & 10 \end{pmatrix}$

b) $BA = \begin{pmatrix} 3 & -2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -11 \\ 18 & 13 \end{pmatrix}$

Hence if we have a matrix A and multiply it by a matrix X, then we need to state whether we want XA or AX as they are often not the same thing. To do this we introduce the terms pre- and post-multiplication. If we **pre-multiply** a matrix A by X we are finding XA, but if we **post-multiply** a matrix A by X we are finding AX.

Identity matrix

Under the operation of multiplication, the identity matrix is one that fulfils the following properties. If A is any matrix and I is the identity matrix, then $A \times I = I \times A = A$. In other words, if any square matrix is pre- or post-multiplied by the identity matrix, then the answer is the original matrix. This is similar to the role that 1 has in multiplication of real numbers, where $1 \times x = x \times 1 = x$ for $x \in \mathbb{R}$.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix for 2×2 matrices and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity matrix for 3×3 matrices.

Only square matrices have an identity of this form.

Zero matrix

Under the operation of addition or subtraction, the matrix that has the identity property is the zero matrix. This is true for a matrix of any size.

For a 2×2 matrix the zero matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and for a 2×3 matrix it is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The role of the zero matrix is similar to the role that 0 has in addition of real numbers, where $0 + x = x + 0 = x$ for $x \in \mathbb{R}$. If we multiply by a zero matrix, the answer will be the zero matrix.

Associativity

Matrix multiplication is associative. This means that $(AB)C = A(BC)$.

We will prove this for 2×2 matrices. The method of proof is the same for any three matrices that will multiply.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and $C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$.

$$\begin{aligned} A(BC) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix}\right] \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} ei + fk & ej + fl \\ gi + hk & gj + hl \end{pmatrix}\right] \\ &= \begin{pmatrix} aei + afk + bgi + bhk & aej + afl + bgj + bhl \\ cei + cfk + dgi + dhk & cej + cfl + dgj + dhl \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (AB)C &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]\begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ &= \left[\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}\right]\begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ &= \begin{pmatrix} aei + bgi + afk + bhk & aej + bgj + afl + + bhl \\ cei + dgi + cfk + + dhk & cej + dgj + cfl + + dhl \end{pmatrix} \\ &= \begin{pmatrix} aei + afk + bgi + bhk & aej + afl + bgj + bhl \\ cei + cfk + dgi + dhk & cej + cfl + dgj + dhl \end{pmatrix} \end{aligned}$$

Hence matrix multiplication on 2×2 matrices is associative.

Distributivity

Matrix multiplication is distributive across addition. This means that $A(B + C) = AB + AC$.

We will prove this for 2×2 matrices. The method of proof is the same for any three matrices that will multiply.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and $C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$.

$$\begin{aligned} A(B + C) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right] \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} e + i & f + j \\ g + k & h + l \end{pmatrix}\right] \\ &= \begin{pmatrix} ae + ai + bg + bk & af + aj + bh + bl \\ ce + ci + dg + dk & cf + cj + dh + dl \end{pmatrix} \end{aligned}$$

$$\begin{aligned} AB + AC &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} + \begin{pmatrix} ai + bk & aj + bl \\ ci + dk & cj + dl \end{pmatrix} \\ &= \begin{pmatrix} ae + ai + bg + bk & af + aj + bh + bl \\ ce + ci + dg + dk & cf + cj + dh + dl \end{pmatrix} \end{aligned}$$

Hence matrix multiplication is distributive over addition in 2×2 matrices.

Exercise 1

1 What is the order of each of these matrices?

a $\begin{pmatrix} 1 & 2 & -3 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 6 & -1 \\ 4 & 3 & 7 \end{pmatrix}$

c $\begin{pmatrix} 2 & 6 & -1 \\ -3 & 3 & 7 \\ 1 & -3 & 2 \end{pmatrix}$ **d** $\begin{pmatrix} 1 \\ k \\ 6k \\ 2 \end{pmatrix}$

2 Alan, Bill and Colin buy magazines and newspapers each week. The tables below show their purchases in three consecutive weeks.

Week 1

	Magazines	Newspapers
Alan	3	1
Bill	2	2
Colin	4	4

Week 2

	Magazines	Newspapers
Alan	1	2
Bill	4	1
Colin	0	1

Week 3

	Magazines	Newspapers
Alan	4	2
Bill	1	0
Colin	1	1

Write each of these in matrix form. What operation do you need to perform on the matrices to find the total number of magazines and the total number of newspapers bought by each of the men? What are these numbers?

3 Simplify these.

a $4\begin{pmatrix} 2 & 1 & 3 \\ 5 & -2 & 3 \\ 7 & -4 & 1 \end{pmatrix}$ **b** $-6\begin{pmatrix} 3 & 4 \\ -1 & 2 \\ -3 & -4 \end{pmatrix}$

c $k\begin{pmatrix} 3 & 6 \\ -4 & -1 \\ 12 & 4 \end{pmatrix}$ **d** $(k - 1)\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

4 Find the unknowns in these equations.

a $\begin{pmatrix} 2 & 4 & -1 \\ 3 & 1 & k \end{pmatrix} = \begin{pmatrix} 2 & 4 & -1 \\ 3 & 1 & 6 \end{pmatrix}$

b $\begin{pmatrix} 2 & 5 \\ 7 & k \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 7 & k^2 \end{pmatrix}$

c $\begin{pmatrix} 2 & 1 \\ 3 & 7 \\ 1 & k^2 \end{pmatrix} = \begin{pmatrix} 2 & k \\ 3 & 7 \\ 1 & k \end{pmatrix}$

d $3\begin{pmatrix} 2 & k \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & k^2 \\ -3 & 9 \end{pmatrix}$

e $\frac{1}{2}\begin{pmatrix} 2 & 3 \\ -1 & \frac{3}{2} \end{pmatrix} = \frac{k}{2}\begin{pmatrix} 4 & 6 \\ -2 & 3k \end{pmatrix}$

f $\begin{pmatrix} 2 & 3 \\ 2 & -k \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ k & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 7 & 6-2k \end{pmatrix}$

g $\begin{pmatrix} 3 & -4 \\ 4 & x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -6 & 1 \end{pmatrix}$

5 For $P = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $R = \begin{pmatrix} -3 & 4 \\ 2 & -3 \\ 4 & 8 \end{pmatrix}$, $S = \begin{pmatrix} 8 & 1 \\ -1 & 4 \end{pmatrix}$,

$T = \begin{pmatrix} 8 & -1 \\ -4 & -3 \\ -4 & 7 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find, if possible:

a $P + Q$ **b** $P + S$ **c** $Q + R$
d $R - T$ **e** $S - U$ **f** $P + S - U$
g $2R + 3T$ **h** $-P - 2S + 3U$

6 Multiply these matrices.

a $\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -3 & 7 \\ 6 & 1 \end{pmatrix}$

b $\begin{pmatrix} 2 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ 8 \end{pmatrix}$

c $\begin{pmatrix} 3 & -4 \\ 2 & -4 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 1 & -2 \end{pmatrix}$

d $\begin{pmatrix} 3 & 5 & -2 \\ 2 & 5 & 2 \\ 1 & -4 & -3 \end{pmatrix}\begin{pmatrix} 2 & 2 & -2 \\ 3 & 7 & -1 \\ -4 & 1 & 0 \end{pmatrix}$

e $\begin{pmatrix} 2 & k \\ -1 & k \end{pmatrix}\begin{pmatrix} 3 & 1 \\ k & 2 \end{pmatrix}$

f $\begin{pmatrix} 1 & 2 & -1 & k \end{pmatrix}\begin{pmatrix} 7 & 0 & k & 2 \\ 3 & -k & 2 & k+1 \\ 6 & 5 & 2k & 3 \\ 3k-4 & 2 & 0 & -2k \end{pmatrix}$

7 If $A = \begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 9 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, find:

a AB **b** $A(BC)$ **c** $(AB)C$ **d** $C(AB)$
e $3BC$ **f** $(A - B)C$ **g** $(2A + B)(A - C)$ **h** $3(A + B)(A - B)$

8 Find the values of x and y.

a $\begin{pmatrix} 2 & x \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & y \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 5 & 3 \end{pmatrix}$

b $\begin{pmatrix} 3 & 7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

c $\begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ y & 4 \end{pmatrix} = \begin{pmatrix} x & 2 \\ -1 & -4 \end{pmatrix}$

d $\begin{pmatrix} 3 & x \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ y & 4 \end{pmatrix} = \begin{pmatrix} 0 & 19 \\ 13 & -11 \end{pmatrix}$

9 If $A = \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix}$ find A^2 and A^3.

10 a The table below shows the number of men, women and children dieting in a school on two consecutive days.

	Men	Women	Children
Day 1	3	2	4
Day 2	5	7	2

Write this in matrix form, calling the matrix A.

b The minimum number of calories to stay healthy is shown in the table below.

	Calories
Men	1900
Women	1300
Children	1100

Write this in matrix form, calling the matrix B.

c Evaluate the matrix AB and explain the result.

d If $C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, and $E = \begin{pmatrix} 1 & 1 \end{pmatrix}$ find:

i EAB
ii AC
iii DB

In each case, explain the meaning of the result.

11 The table below shows the numbers of games won, drawn and lost for five soccer teams.

	Won	Drawn	Lost
Absolutes	3	4	7
Brilliants	6	2	6
Charismatics	10	1	3
Defenders	3	9	2
Extras	8	3	3

a Write this as matrix P.

b If a team gains 3 points for a win, 1 point for a draw and no points for losing, write down a matrix Q that when multiplied by P will give the total points for each team.

c Find this matrix product.

12 Given that $N = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}$ find:

a $2N - 3I$ **b** $N^2 - 2I$ **c** $N^2 - 3N + 2I$

13 If $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} m & 2 \\ n & -3 \end{pmatrix}$ find the values of m and n such that the multiplication of A and B is commutative.

14 If $M = \begin{pmatrix} 4 & 2 \\ 1 & -3 \end{pmatrix}$ and $M^2 - M - 4\mathrm{I} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, find the value of k.

15 If $M^2 = 2M + \mathrm{I}$ where M is any 2×2 matrix, show that $M^4 = 2M^2 + 8M + 5\mathrm{I}$.

16 Given that $A = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ and $A^2 - 6A + c\mathrm{I} = 0$, find the value of c.

17 If $A = \begin{pmatrix} 1 & 3 & c \\ 2 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & c \\ 3 & 0 \\ -c & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & -6 \\ 2 & \frac{1}{2} \end{pmatrix}$, find the value of c such that $AB = 2C$.

18 If $P = \begin{pmatrix} 1 & 4 & -3 \\ c & 2 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & 1 \\ 2 & 0 \\ 3 & c \end{pmatrix}$, find the products PQ and QP.

19 Find the values of x and y for which the following pairs of matrices are commutative.

a $X = \begin{pmatrix} x & 2 \\ 1 & -2 \end{pmatrix}$ and $Y = \begin{pmatrix} 3 & y \\ 5 & 1 \end{pmatrix}$

b $X = \begin{pmatrix} 3 & y \\ x & -2 \end{pmatrix}$ and $Y = \begin{pmatrix} -1 & y \\ 2 & 1 \end{pmatrix}$

c $X = \begin{pmatrix} 8 & y \\ 3 & -2 \end{pmatrix}$ and $Y = \begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$

20 Find in general form the 2×2 matrix A that commutes with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

11.2 Determinants and inverses of matrices

Finding inverse matrices

If we think of a matrix A multiplied by a matrix B to give the identity matrix, then the matrix B is called the **inverse** of A and is denoted by A^{-1}. If A is any matrix and I is the identity matrix, then the inverse fulfils the following property:

$$A \times A^{-1} = A^{-1} \times A = \mathrm{I}$$

Consider a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let its inverse be $A^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Equating elements: $ap + br = 1$ (i)

$aq + bs = 0$ (ii)

$cp + dr = 0$ (iii)

$cq + ds = 1$ (iv)

(i) $\times\, d \Rightarrow adp + bdr = d$ (v)

(ii) $\times\, b \Rightarrow bcp + bdr = 0$ (vi)

(v) $-$ (vi) $\Rightarrow adp - bcr = d$

$$\Rightarrow p = \frac{d}{ad - bc}$$

Using a similar method we find

$$q = \frac{-b}{ad - bc}$$

$$r = \frac{-c}{ad - bc}$$

$$\text{and } s = \frac{a}{ad - bc}$$

$$\text{so } A^{-1} = \begin{pmatrix} \dfrac{d}{ad - bc} & q = \dfrac{-b}{ad - bc} \\ q = \dfrac{-c}{ad - bc} & q = \dfrac{a}{ad - bc} \end{pmatrix}$$

$$= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

> $ad - bc$ is known as the **determinant** of a 2×2 matrix.

The notation for the determinant of matrix A is Det(A) or $|A|$ and for the matrix above is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Provided the determinant does not equal zero, the matrix has an inverse. A matrix where $ad - bc = 0$ is called a **singular matrix** and if $ad - bc \neq 0$ then it is called a **non-singular matrix**.

Example

Find the determinant of the matrix $B = \begin{pmatrix} 3 & 1 \\ 2 & 6 \end{pmatrix}$.

$\text{Det}(B) = (3)(6) - (2)(1) = 16$

Method to find the inverse of a 2×2 matrix

If a calculator cannot be used then:

1. Evaluate the determinant to check the matrix is non-singular and divide each element by the determinant.
2. Interchange the elements a and d in the leading diagonal.
3. Change the signs of the remaining elements b and c.

On a calculator paper where no variables are involved, a calculator should be used.

Example

Find the inverse of $M = \begin{pmatrix} 3 & 7 \\ -2 & -5 \end{pmatrix}$.

$\text{Det}(M) = -15 - (-14) = -1$ so M^{-1} exists.

$$M^{-1} = \frac{1}{-1}\begin{pmatrix} -5 & -7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ -2 & -3 \end{pmatrix}$$

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It is not part of this syllabus to find the inverse of a 3×3 matrix by hand, but you need to be able to do this on a calculator, and you need to be able to verify that a particular matrix is the inverse of a given matrix.

Example

Find the inverse of $\begin{pmatrix} 1 & 3 & 1 \\ 3 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$.

On a calculator this appears as:

```
MATRIX[A] 3 ×3
[1      3     1     ]
[3      -1    1     ]
[1      -2    1     ]



3,3=1
```

and the answer is:

```
[A]-1
 [[-.1  .5   -.4]
  [.2   0    -.2]
  [.5   -.5  1  ]]
```

Example

Verify that $\begin{pmatrix} \frac{4}{7} & \frac{-6}{7} & \frac{-11}{7} \\ \frac{3}{7} & \frac{-8}{7} & \frac{-10}{7} \\ \frac{-1}{7} & \frac{5}{7} & \frac{8}{7} \end{pmatrix}$ is the inverse of $\begin{pmatrix} 2 & 1 & 4 \\ 2 & -3 & -1 \\ -1 & 2 & 2 \end{pmatrix}$.

If we multiply these together then the answer is:

```
[A]*[B]
        [[1 0 0]
         [0 1 0]
         [0 0 1]]
```

Since this is the identity matrix, the matrices are inverses of each other.

Example

If $A = \begin{pmatrix} 0 & 0 & 2m \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ n & 0 & 0 \end{pmatrix}$, find the values of m and n.

$$\begin{pmatrix} 0 & 0 & 2m \\ 0 & 1 & 0 \\ m & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ n & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2mn & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so $m = 1$

and $2mn = 1 \Rightarrow n = \frac{1}{2}$

General results for inverse matrices

If $AB = I$ then $B = A^{-1}$ and $A = B^{-1}$.

That is, the matrices are inverses of each other.

Proof

Let $AB = I$.

If we pre-multiply both sides of the equation by A^{-1} then

$$(A^{-1}A)B = A^{-1}I$$
$$\Rightarrow IB = A^{-1}I$$
$$\Rightarrow B = A^{-1}$$

Similarly if we post-multiply both sides of the equation by B^{-1} then

$$A(BB^{-1}) = IB^{-1}$$
$$\Rightarrow AI = IB^{-1}$$
$$\Rightarrow A = B^{-1}$$

$(AB)^{-1} = B^{-1}A^{-1}$

Proof

We begin with $B^{-1}A^{-1}$.

Pre-multiplying by AB:

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1}$$
$$= AIA^{-1}$$
$$= AA^{-1}$$
$$= I$$

Therefore the inverse of AB is $B^{-1}A^{-1}$, so
$(AB)^{-1} = B^{-1}A^{-1}$

Finding the determinant of a 3×3 matrix

This is done by extracting the 2×2 determinants from the 3×3 determinant, and although it can be done on a calculator very easily, it is important to know how to do this by hand.

3×3 determinants are particularly important when we come to work with vector equations of planes in Chapter 13.

Method

If the row and column through a particular entry in the 3×3 determinant are crossed out, four entries are left that form a 2×2 determinant, and this is known as the 2×2 determinant through that number. However, there is a slight complication. Every entry in a 3×3 determinant has a sign associated with it, which is not the sign of the entry itself. These are the signs:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Therefore in the determinant $\begin{vmatrix} 6 & 2 & 3 \\ 4 & -1 & -2 \\ -3 & 0 & 5 \end{vmatrix}$ the 2×2 determinant through 6 is $\begin{vmatrix} -1 & -2 \\ 0 & 5 \end{vmatrix}$.

In the same determinant $\begin{vmatrix} 6 & 2 & 3 \\ 4 & -1 & -2 \\ -3 & 0 & 5 \end{vmatrix}$, the 2×2 determinant through 4 is $-\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$.

To find the determinant of a 3×3 matrix, we extract three 2×2 determinants and then evaluate these as before. It is usual to extract the 2×2 determinants from the top row of the 3×3 determinant.

Example

Without using a calculator, evaluate $\begin{vmatrix} 1 & 6 & 9 \\ 2 & -2 & 5 \\ -8 & 1 & 4 \end{vmatrix}$.

$$\begin{vmatrix} 1 & 6 & 9 \\ 2 & -2 & 5 \\ -8 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} -2 & 5 \\ 1 & 4 \end{vmatrix} - 6\begin{vmatrix} 2 & 5 \\ -8 & 4 \end{vmatrix} + 9\begin{vmatrix} 2 & -2 \\ -8 & 1 \end{vmatrix}$$
$$= (-8 - 5) - 6(8 + 40) + 9(2 - 16)$$
$$= -13 - 288 - 126 = -427$$

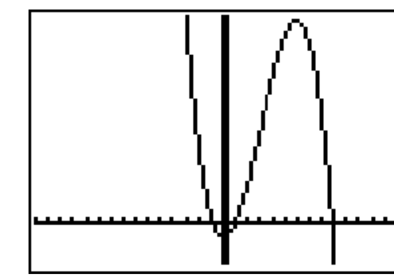
Example

Find the values of y for which the matrix $M = \begin{pmatrix} 1 & 4 & 2 \\ -3 & 2y & 9 \\ y^2 & 1 & -1 \end{pmatrix}$ is singular.

For the matrix to be singular $\text{Det}(M) = 0$.

$$\Rightarrow \begin{vmatrix} 2y & 9 \\ 1 & -1 \end{vmatrix} - 4\begin{vmatrix} -3 & 9 \\ y^2 & -1 \end{vmatrix} + 2\begin{vmatrix} -3 & 2y \\ y^2 & 1 \end{vmatrix} = 0$$
$$\Rightarrow (-2y - 9) - 4(3 - 9y^2) + 2(-3 - 2y^3) = 0$$
$$\Rightarrow -4y^3 + 36y^2 - 2y - 27 = 0$$

We solve this using a calculator.



$\Rightarrow y = -0.805,\ 0.947$ or 8.86

General results for determinants

$$\text{Det}(AB) = \text{Det}(A) \times \text{Det}(B)$$

This is a useful result, which can save time, and is proved below for 2×2 matrices. The proof for 3×3 matrices would be undertaken in exactly the same way.

Proof

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

$$AB = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

$$\text{Det}(AB) = (ap + br)(cq + ds) - (aq + bs)(cp + dr)$$
$$= acpq + adps + bcqr + bdrs - acpq - adqr - bcps - bdrs$$
$$= adps + bcqr - adqr - bcps$$

Now $\text{Det}(A) = ad - bc$ and $\text{Det}(B) = ps - qr$

So $\text{Det}(A) \times \text{Det}(B) = (ad - bc)(ps - qr)$
$= adps - adqr - bcps + bcqr$

Hence $\text{Det}(AB) = \text{Det}(A) \times \text{Det}(B)$

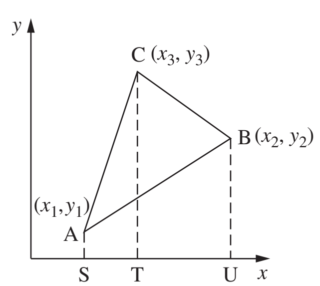
Example

If $A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -4 & 3 \\ 6 & 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -4 \\ 3 & 7 & 2 \\ -1 & 4 & 3 \end{pmatrix}$, find $\text{Det}(AB)$. If possible find $(AB)^{-1}$.

Using a calculator, $\text{Det}(A) = 0$ and $\text{Det}(B) = -85$. Hence $\text{Det}(AB) = 0$. Since $\text{Det}(AB) = 0$, the matrix is singular so AB has no inverse.

The area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$.

Proof



Considering the diagram shown above:

$$\begin{aligned}
\text{Area ABC} &= \text{area SATC} + \text{area TCBU} - \text{area SABU} \\
&= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_2 + y_3)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1) \\
&= \frac{1}{2}(x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 + x_2y_2 + x_2y_3 - x_3y_2 - x_3y_3 - x_2y_1 \\
&\quad - x_2y_2 + x_1y_1 + x_1y_2) \\
&= \frac{1}{2}(x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1) \\
&= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}
\end{aligned}$$

If A, B and C lie on a straight line (are collinear), then the area of the triangle is zero, and this is one possible way to show that three points are collinear.

Example

Without using a calculator, find the area of the triangle PQR whose vertices have coordinates $(1, 2)$, $(2, -3)$ and $(5, -1)$.

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & -3 & -1 \end{vmatrix} \\
&= \frac{1}{2}[(-2 + 15) - (-1 - 10) + (-3 - 4)] \\
&= \frac{1}{2}(13 + 11 + 7) = \frac{31}{2} \text{ units}^2
\end{aligned}$$

Exercise 2

1 Find the inverse of each of these matrices.

a $\begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 7 \\ -3 & 10 \end{pmatrix}$ **c** $\begin{pmatrix} 4 & 8 \\ -9 & 1 \end{pmatrix}$

d $\begin{pmatrix} 3 & 8 & -1 \\ 2 & 5 & 3 \\ 8 & -4 & 2 \end{pmatrix}$ **e** $\begin{pmatrix} 6 & 5 & 7 \\ -9 & 2 & 6 \\ 0 & 5 & -3 \end{pmatrix}$ **f** $\begin{pmatrix} 4 & 6 & -6 \\ 2 & 5 & 1 \\ 8 & 3 & 8 \end{pmatrix}$

g $\begin{pmatrix} k & 5 \\ 3k & 1 \end{pmatrix}$ **h** $\begin{pmatrix} 3k & k \\ 2k - 1 & k + 2 \end{pmatrix}$

2 If $P = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, $Q = \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 3 & -3 \\ 2 & 4 \end{pmatrix}$, $S = \begin{pmatrix} 4 & 7 \\ 9 & 1 \end{pmatrix}$, $PX = Q$, $QY = R$ and $RZ = S$, find the matrices X, Y and Z.

3 Evaluate these determinants.

a $\begin{vmatrix} 4 & 7 \\ 6 & 3 \end{vmatrix}$ **b** $\begin{vmatrix} 6 & -1 \\ -4 & 8 \end{vmatrix}$

c $\begin{vmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & 1 & 4 \end{vmatrix}$ **d** $\begin{vmatrix} 2 & 5 & -3 \\ 7 & 3 & 6 \\ 2 & 1 & -5 \end{vmatrix}$

4 Expand and simplify these.

a $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$ **b** $\begin{vmatrix} \sin\theta & \cos\theta \\ \sin\theta & -\cos\theta \end{vmatrix}$ **c** $\begin{vmatrix} x + 1 & x - 1 \\ x & x - 2 \end{vmatrix}$

d $\begin{vmatrix} 0 & a & c \\ a & 0 & b \\ c & b & 0 \end{vmatrix}$ **e** $\begin{vmatrix} \cos\theta & \sin\theta & \tan\theta \\ \tan\theta & \cos\theta & \sin\theta \\ \sin\theta & \tan\theta & \cos\theta \end{vmatrix}$ **f** $\begin{vmatrix} y - 1 & 0 & y + 1 \\ 1 & 1 & -1 \\ y + 1 & 1 & 1 \end{vmatrix}$

g $\begin{vmatrix} 1 & 1 & 1 \\ b^2 & a^2 & a^2 \\ a & b & a \end{vmatrix}$

5 Using determinants, find the area of the triangle PQR, where P, Q and R are the points:

a $(1, 4)$, $(3, -2)$, $(4, -1)$

b $(3, 7)$, $(-4, -4)$, $(1, 5)$

c $(-3, 5)$, $(9, 1)$, $(5, -1)$

6 Using determinants, determine whether each set of points is collinear.

a $(1, 2), (6, 7), (3, 4)$

b $(1, 3), (5, -2), (7, -3)$

c $(2, 3), (5, 18), (-3, -22)$

7 Verify that these matrices are the inverses of each other.

a $\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-7}{3} & \frac{-1}{3} & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ 6 & -2 & 22 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & 1 & \frac{-1}{4} \\ \frac{-5}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{1}{8} \end{pmatrix}$

c $\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 2 \\ -1 & 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{-7}{16} & \frac{3}{8} & \frac{-5}{16} \\ \frac{-5}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{13}{16} & \frac{-1}{8} & \frac{7}{16} \end{pmatrix}$

8 Find the value of k for which the matrix $\begin{pmatrix} k^2 + k + 2 & k^2 & 0 \\ k + 4 & 2 & k^2 \\ 1 & 1 & 1 \end{pmatrix}$ is singular.

9 If $A = \begin{pmatrix} 1 & 3 \\ k & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -2 \\ 1 & -k \end{pmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$.

10 Find the values of c for which the matrix $\begin{pmatrix} 4 & -2 & 6 \\ 1 & c & 9 \\ 0 & 3 & c \end{pmatrix}$ is singular.

11 Find the values of y such that $\begin{vmatrix} y & 2y \\ y & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 2 & 0 \\ 1 & 5 & 1 \end{vmatrix}$.

12 M is the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & x & 0 \\ x^2 & 0 & x \end{pmatrix}$ and N is the matrix $\begin{pmatrix} -y & 0 & 1 \\ 1 & y & -y \\ y^2 & 0 & 0 \end{pmatrix}$. By evaluating the product MN, find the values of x and y for which M is the inverse of N.

13 If $2A - 3BX = B$, where A, B and X are 2×2 matrices, find

a X in terms of A and B

b X given that $B^{-1}A = 2I$, where I is the identity matrix.

14 The matrix $M = \begin{pmatrix} x - 3 & x - 1 \\ x + 1 & x + 3 \end{pmatrix}$.

a Show that $\mathrm{Det}(M)$ is independent of x.

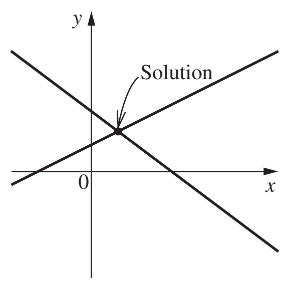
b Find M^{-1}.

11.3 Solving simultaneous equations in two unknowns

The techniques of solving two simultaneous equations in two unknowns have been met before, but it is worth looking at the different cases and then examining how we can use matrices to solve these.

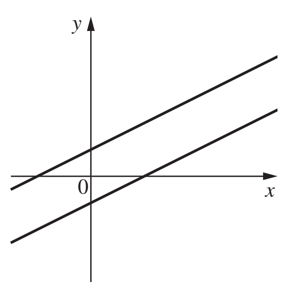
When we have two linear equations there are three possible scenarios, which are shown in the diagrams below.

The lines intersect, giving a unique solution



In this case solving the pair of simultaneous equations using a method of elimination or substitution will give the unique solution.

The lines are parallel, giving no solution



This occurs when we attempt to eliminate a variable and find we have a constant equal to zero.

Example

Determine whether the following equations have a solution.

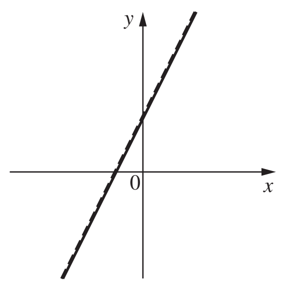
$x + 5y = 7$ equation (i)

$-3x - 15y = 16$ equation (ii)

$3(\text{i}) + (\text{ii}) \Rightarrow 0 = 23$

This is not possible, so there is no solution.

The lines are the same, giving infinite solutions



The lines are coincident.

In this case, when we try to eliminate a variable we find $0 = 0$, but we can actually give a solution.

Example

Find the solution to:

$4x - y = 5$ equation (i)
$16x - 4y = 20$ equation (ii)

If we do 4(i) – (ii) we find $0 = 0$.

Hence the solution can be written as $y = 4x - 5$.

If we are asked to show that simultaneous equations are consistent, this means that they either have a unique solution or infinite solutions. If they are inconsistent, they have no solution.

Using matrices to solve simultaneous equations in two unknowns

This is best demonstrated by example.

Example

Find the solution to these simultaneous equations.

$3x + 2y = 4$
$x - y = 5$

This can be represented in matrix form as

$$\begin{pmatrix}3 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$$

To solve this we pre-multiply both sides by the inverse matrix. This can be calculated either by hand or by using a calculator.

Hence $$\begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\\frac{1}{5} & \frac{-3}{5}\end{pmatrix}\begin{pmatrix}3 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\\frac{1}{5} & \frac{-3}{5}\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\\frac{1}{5} & \frac{-3}{5}\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{14}{5}\\\frac{-11}{5}\end{pmatrix}$$

So $x = \frac{14}{5}, y = -\frac{11}{5}$

If we are asked to find when equations have no unique solution, then we would need to show the matrix is singular. However, if we need to distinguish between the two cases here, i.e. no solution or infinite solutions, then we need to use Gaussian elimination.

Example

Show that the following system of equations does not have a unique solution.

$2x - 3y = 7$
$6x - 9y = 20$

This can be represented in matrix form as

$$\begin{pmatrix}2 & -3\\6 & -9\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}7\\20\end{pmatrix}$$

If $A = \begin{pmatrix}2 & -3\\6 & -9\end{pmatrix}$ then $\text{Det}(A) = -18 + 18 = 0$.

Hence the matrix is singular and the system of equations does not have a unique solution.

We can also write the simultaneous equations as what is called an **augmented matrix** and solve from here. This is effectively a neat way of representing elimination, but becomes very helpful when we deal with three equations in three unknowns. We will demonstrate this in the same example.

Example

The augmented matrix looks like this.

$$\left(\begin{array}{cc|c}3 & 2 & 4\\1 & -1 & 5\end{array}\right)$$

This is called Gaussian elimination or row reduction.

We now conduct row operations on the augmented matrix to find a solution.

Changing Row 1 to Row 1 − 3 (Row 2)

We are trying to make the first element zero.

$$\Rightarrow \left(\begin{array}{cc|c}0 & 5 & -11\\1 & -1 & 5\end{array}\right)$$

This is the same as

$$\begin{pmatrix}0 & 5\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-11\\5\end{pmatrix}$$

So the first row gives:

$5y = -11$

$\Rightarrow y = -\frac{11}{5}$

We now substitute in the second row:

$$x - y = 5$$
$$\Rightarrow x + \frac{11}{5} = 5$$
$$\Rightarrow x = \frac{14}{5}$$

Example

Determine whether the following set of equations has no solution or infinite solutions.

$$x - 3y = 7$$
$$-3x + 9y = 15$$

The augmented matrix for these equations is

$$\left(\begin{array}{cc|c} 1 & -3 & 7 \\ -3 & 9 & 15 \end{array}\right)$$

Changing Row 1 to 3 (Row 1) + Row 2

$$\Rightarrow \left(\begin{array}{cc|c} 0 & 0 & 36 \\ -3 & 9 & 15 \end{array}\right)$$

Hence we have a case of $0 = 36$, which is inconsistent, so the equation has no solution.

Had the equation had infinite solutions we would have had a whole line of zeros.

Exercise 3

1 Use the inverse matrix to solve these equations.

a $3x + y = 4$, $x - 2y = 7$ **b** $3p - 5q = 7$, $p - 2q = 7$

c $3y + 1 - 2x = 0$, $4x + 3y - 4 = 0$ **d** $y = 3x - 4$, $3y = 7x + 2$

2 Using a method of row reduction, solve these pairs of simultaneous equations.

a $x - 5y = 7$, $3x + 5y = 10$ **b** $a - 3b = 8$, $2a + 5b = 7$

c $2x + 3y - 8 = 0$, $x - 2y + 7 = 0$ **d** $y = \frac{3}{2}x - 9$, $y = 4x - 1$

3 Use the method of inverse matrices or row reduction to uniquely solve the following pairs of simultaneous equations. In each case state any restrictions there may be on the value of k.

a $(2k + 1)x - y = 1$, $(k + 1)x - 2y = 3$ **b** $x - ky = 3$, $kx - 3y = 3$

c $y + (2k - 1)x - 1 = 0$, $5y - kx + 7 = 0$ **d** $y = kx - 4$, $(k + 1)y = -3x + 10$

4 By evaluating the determinant, state whether the simultaneous equations have a unique solution.

a $3x + 2y = -7$, $6x - 4y = 14$ **b** $8x + 7y = 15$, $3x - 8y = 13$

c $y = 2x - 5$, $2y - 4x = -10$ **d** $(3k - 1)y - x = 5$, $4y - (k + 1)x = 11$

5 Determine the value of c for which the simultaneous equations have no solution. What can you say about the lines in each case?

a $cy + (3c - 1)x = 7$, $cy = -2x + 3c$ **b** $cx - (2c - 4)y = 15$, $(c + 1)x - 2cy = 9$

6 State with a reason which of these pairs of equations are consistent.

a $y - 3x = 7$, $\frac{y}{2} - 5x = 9$ **b** $2y - 3x = -7$, $y = \frac{3}{2}x - \frac{14}{4}$ **c** $y + 2x - 3 = 0$, $x = -\frac{y}{2} + \frac{6}{4}$

7 Find the value of p for which the lines are coincident.

$$2x - 4y - 2p = 0$$
$$px - 6y - 9 = 0$$
$$x - 2y - p = 0$$

8 Find the value of λ for which the equations are consistent and in this case find the corresponding values of y and x.

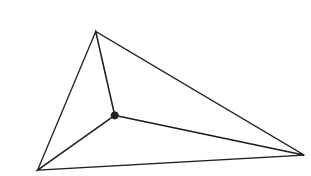
$$4x + \lambda y = 10$$
$$3x - y = 4$$
$$4x + 6y = -2$$

11.4 Solving simultaneous equations in three unknowns

We will see in Chapter 13 that an equation of the form $ax + by + cz = d$ is the equation of a plane. Because there are three unknowns, to solve these simultaneously we need three equations. It is important at this stage to consider various scenarios, which, like lines, lead to a unique solution, infinite solutions or no solution.

Unique solution

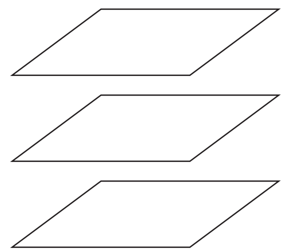
The three planes intersect in a point.



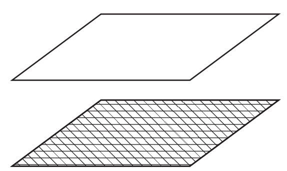
In this case solving the three equations simultaneously using any method will give the unique solution.

No solution

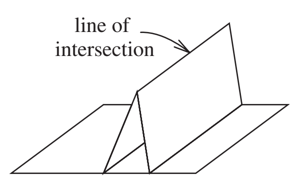
The three planes are parallel.



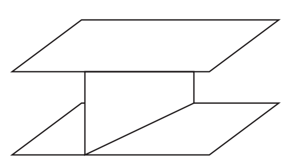
Two planes are coincident and the third plane is parallel.



Two planes meet in a line and the third plane is parallel to the line of intersection

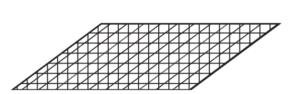


Two planes are parallel and the third plane cuts the other two.

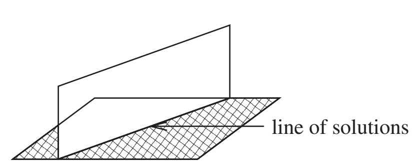


Infinite solutions

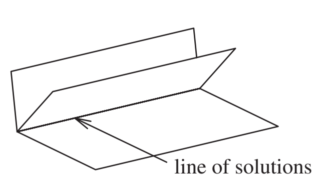
Three planes are coincident. In this case the solution is a plane of solutions.



Two planes are coincident and the third plane cuts the other two in a line. In this case the solution is a line of solutions.



All three meet in a common line. In this case the solution is a line of solutions.



How do we know which case we have? This works in exactly the same way as a pair of simultaneous equations in two unknowns when we consider the augmented matrix.

If there are four zeros in a row of the augmented matrix there are infinite solutions. It does not matter which row it is. Therefore the augmented matrix $\begin{pmatrix} 3 & 2 & -1 & | & 4 \\ 4 & 2 & 6 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$ would produce infinite solutions.

If there are three zeros in a row of the augmented matrix there are no solutions. Again it does not matter which row it is, but the three zeros have to be the first three entries in the row. Therefore the augmented matrix $\begin{pmatrix} 3 & 2 & -1 & | & 4 \\ 4 & 2 & 6 & | & 1 \\ 0 & 0 & 0 & | & 4 \end{pmatrix}$ would produce no solution.

All other augmented matrices will produce a unique solution. For example $\begin{pmatrix} 3 & 2 & -1 & | & 4 \\ 4 & 2 & 6 & | & 1 \\ 0 & 4 & 0 & | & 0 \end{pmatrix}$ has a unique solution even though there is one row with three zeros in it. This highlights the fact that the position of the three zeros is important.

Elimination

To do this we eliminate one variable using two pairs of equations, leaving us with a pair of simultaneous equations in two unknowns.

Solving simultaneous equations in three unknowns

We are often told which method to use, but if not:

1 If a unique solution is indicated, any method can be used to find it.
2 If we want to distinguish between unique and non-unique solutions, then checking whether the matrix is singular is the easiest method.
3 If we want to establish that there is no solution or find the infinite solutions, then row operations are usually the easiest.

Example

Solve these equations.

$4x + 2y + z = 0$ equation (i)
$3x - 7y - 2z = 20$ equation (ii)
$x + y + 4z = 6$ equation (iii)

$2(\text{i}) + (\text{ii}) \Rightarrow 11x - 3y = 20$ equation (iv)
$2(\text{ii}) + (\text{iii}) \Rightarrow 7x - 13y = 46$ equation (v)

$13(\text{iv}) - 3(\text{v}) \Rightarrow 122x = 122$
$\Rightarrow x = 1$

Substitute in equation (iv):
$11 - 3y = 20$
$\Rightarrow y = -3$

Substituting x and y in equation (iii):
$1 - 3 + 4z = 6$
$\Rightarrow z = 2$

Substitution

To do this we make one variable the subject of one equation, substitute this in the other two equations and then solve the resulting pair in the usual way.

Example

Solve these equations.

$$3x + 4y - z = -2 \text{ equation (i)}$$
$$2x + 5y + 2z = 7 \text{ equation (ii)}$$
$$x - 3y - z = 1 \text{ equation (iii)}$$

Rearranging equation (i) gives $z = 3x + 4y + 2$.

Substituting in equation (ii):

$$2x + 5y + 6x + 8y + 4 = 7$$
$$\Rightarrow 8x + 13y = 3 \text{ equation (iv)}$$

Substituting in equation (iii):

$$x - 3y - 3x - 4y - 2 = 1$$
$$\Rightarrow -2x - 7y = 3 \text{ equation (v)}$$

Rearranging equation (iv) gives $x = \dfrac{3 - 13y}{8}$ equation (vi)

Substituting in equation (v):

$$-2\left(\frac{3 - 13y}{8}\right) - 7y = 3$$
$$\Rightarrow -6 + 26y - 56y = 24$$
$$\Rightarrow y = -1$$

Substituting in equation (vi):

$$\Rightarrow \quad x = \frac{3 + 13}{8} = 2$$

Substituting in equation (iii):

$$2 + 3 - z = 1$$
$$\Rightarrow z = 4$$

Using inverse matrices

In this case we write the equations in matrix form and then multiply each side of the equation by the inverse matrix. In other words, if $AX = B$ then $X = A^{-1}B$.

Remember that we must pre-multiply by A^{-1}.

Example

Solve these equations using the inverse matrix.

$$2x + y - 3z = -6$$
$$x + y + 4z = 19$$
$$2x + y - 5z = -14$$

Writing these equations in matrix form:

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & 1 & 4 \\ 2 & 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 19 \\ -14 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4.5 & -1 & -3.5 \\ -6.5 & 2 & 5.5 \\ 0.5 & 0 & -0.5 \end{pmatrix}\begin{pmatrix} -6 \\ 19 \\ -14 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

As with lines, if the matrix is singular, then the system of equations has no solution or infinite solutions. Further work using one of the other methods is necessary to distinguish between the two cases.

Example

Does the following system of equations have a unique solution?

$$x + 3y - 4z = 2$$
$$2x - y + 5z = 1$$
$$3x - 5y + 14z = 7$$

Writing these equations in matrix form:

$$\begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & 5 \\ 3 & -5 & 14 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$$

If we let $A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & 5 \\ 3 & -5 & 14 \end{pmatrix}$ then $\mathrm{Det}(A) = -140$ (using a calculator) and hence the system of equations has a unique solution.

Using row operations

The aim is to produce as many zeros in a row as possible.

Example

Find the unique solution to this system of equations using row operations.

$$x - 3y + 2z = -3$$
$$2x + 4y - 3z = 11$$
$$x + y + 2z = 1$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & -3 \\ 2 & 4 & -3 & 11 \\ 1 & 1 & 2 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & -3 \\ 2 & 4 & -3 & 11 \\ 0 & -2 & 7 & -9 \end{array}\right)$$

Change Row 3 to 2 (Row 3) − Row 2.

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & -3 \\ 0 & 10 & -7 & 17 \\ 0 & -2 & 7 & -9 \end{array}\right)$$

Change Row 2 to Row 2 − 2 (Row 1).

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & -3 \\ 0 & 10 & -7 & 17 \\ 0 & 0 & 28 & -28 \end{array}\right)$$

Change Row 3 to 5 (Row 3) + Row 2.

We always begin by producing a pair of zeros above each other in order that when we carry out more row operations on those two lines, we do not go around in circles creating zeros by eliminating ones we already have. Hence the final row operation must use Rows 2 and 3.

We cannot produce any more zeros, so the equation will have a unique solution and we now solve using the rows.

From Row 3

$28z = -28$

$\Rightarrow z = -1$

From Row 2

$10y - 7z = 17$

$\Rightarrow y = 1$

From Row 1

$x - 3y + 2z = -3$

$x - 3 - 2 = -3$

$\Rightarrow x = 2$

However, the strength of row operations is in working with infinite solutions and no solution.

In the case of no solution, row operations will produce a line of three zeros.

Example

Verify that this system of equations has no solution.

$$\begin{aligned} x - 3y + 4z &= 5 \\ 2x - y + 3z &= 7 \\ 3x - 9y + 12z &= 15 \end{aligned}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & -3 & 4 & 5 \\ 2 & -1 & 3 & 7 \\ 3 & -9 & 12 & 14 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 4 & 5 \\ 2 & -1 & 3 & 7 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Change Row 3 to Row 3 − 3 (Row 1).

Since there is a line of three zeros, the system of equations is inconsistent and has no solution.

In the case of infinite solutions we will get a line of four zeros, but as we saw previously there are two possibilities for the solution. In Chapter 13 the format of these solutions will become clearer, but for the moment, if a line of solutions is given, the answer will be dependent on one parameter. A plane of solutions can occur only if the three planes are coincident – that is, the three equations are actually the same – and in this case the equation of the plane is actually the solution.

Example

Show that this system of equations has infinite solutions and find the general form of these solutions.

$$\begin{aligned} 2x + y - 3z &= 1 \\ 2x + 2y - 4z &= 5 \\ 6x + 3y - 9z &= 3 \end{aligned}$$

The augmented matrix is: $\left(\begin{array}{ccc|c} 2 & 1 & -3 & 1 \\ 2 & 2 & -4 & 5 \\ 6 & 3 & -9 & 3 \end{array}\right)$

$$\Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 2 & -4 & 5 \\ 6 & 3 & -9 & 3 \end{array}\right)$$

Change Row 1 to 3 (Row 1) − Row 3.

Hence the system has infinite solutions. We now eliminate one of the other variables.

$$\Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 7 \\ 6 & 3 & -9 & 3 \end{array}\right)$$

Change Row 2 to 2 (Row 2) − Row 3.

This can be read as $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 6 & 3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 3 \end{pmatrix}$.

By letting $y = \lambda$ it is clear that we can get the solutions for x and z in terms of λ.

Hence if we let $y = \lambda$, then $\lambda + z = 7 \Rightarrow z = 7 - \lambda$.

If we now substitute these into the equation $6x + 3y - 9z = 3$ we will find x.

$$\begin{aligned} &6x + 3\lambda - 9(7 - \lambda) = 3 \\ \Rightarrow &6x + 3\lambda - 63 + 9\lambda = 3 \\ &\Rightarrow x = 11 - 2\lambda \end{aligned}$$

Hence the general solution to the equation is $x = 11 - 2\lambda$, $y = \lambda$, $z = 7 - \lambda$.

This is the parametric equation of a line, and we will learn in Chapter 13 how to write this in other forms.

This is a line of solutions. By looking at the original equations we can see that the third equation is three times the first equation, and hence this is a case of two planes being coincident and the third plane cutting these two in a line.

Example

Show that the following system of equations has infinite solutions and find the general form of these solutions.

$$\begin{aligned} x + y - 3z &= 2 \\ 2x + 2y - 6z &= 4 \\ 4x + 4y - 12z &= 8 \end{aligned}$$

The augmented matrix is: $\left(\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 2 & 2 & -6 & 4 \\ 4 & 4 & -12 & 8 \end{array}\right)$

$$\Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 2 & -6 & 4 \\ 4 & 4 & -12 & 8 \end{array}\right)$$

Change Row 1 to 2 (Row 1) − Row 2.

Hence it is clear that the system has infinite solutions. In this case we cannot eliminate another variable, because any more row operations will eliminate all the variables.

This can be read as $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & -6 \\ 4 & 4 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 8 \end{pmatrix}$.

If we look back at the original equations, we can see that they are in fact the same equation, and hence we have the case of three coincident planes, which leads to a plane of solutions. The plane itself is the solution to the equations, i.e. $x + y - 3z = 2$.

Example

Determine what type of solutions the following system of equations has, and explain the arrangement of the three planes represented by these equations.

$$\begin{aligned} 2x + 3y - 2z &= 1 \\ 4x + 6y - 4z &= 2 \\ 6x + 9y - 6z &= 4 \end{aligned}$$

The augmented matrix for this system is $\left(\begin{array}{ccc|c} 2 & 3 & -2 & 1 \\ 4 & 6 & -4 & 2 \\ 6 & 9 & -6 & 4 \end{array}\right)$.

In this case if we perform row operations we get conflicting results.

Changing Row 1 to 2 (Row 1) − Row 2 gives $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 6 & -4 & 2 \\ 6 & 9 & -6 & 4 \end{array}\right)$, which implies the system has infinite solutions.

However, changing Row 1 to 3 (Row 1) − Row 3 gives $\left(\begin{array}{ccc|c} 0 & 0 & 0 & -1 \\ 4 & 6 & -4 & 2 \\ 6 & 9 & -6 & 4 \end{array}\right)$, which implies the system has no solution.

If we now look back at the equations we can see that the first and second equations are multiples of each other. In the third equation the coefficients of x, y and z are multiples of those coefficients in the first and second equations. This means we have two coincident planes and a parallel plane.

Hence the system actually has no solution. This is also obvious from the initial equations since we clearly do not have three coincident planes.

The reasoning behind this will be explained in Chapter 13.

Row operations on a calculator

A calculator is capable of doing this, but there are a number of points that need to be made. Obviously, if the question appears on a non-calculator paper then this is not an option. However, if a calculator is allowed then it is useful. To find a unique solution we put the 3×4 augmented matrix into the calculator as usual.

Example

Use a calculator to find the solution to this system of equations.

$$\begin{aligned} x + 2y + z &= 3 \\ 3x + y - z &= 2 \\ x + 4y + 2z &= 4 \end{aligned}$$

The augmented matrix for this system is $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & 2 & 4 \end{array}\right)$

The calculator display is shown below:

```
rref([A])
    [[1 0 0 2 ]
     [0 1 0 -1]
     [0 0 1 3 ]]
```

This can be read as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Hence $x = 2$, $y = -1$, and $z = 3$.

If we put in a system of equations that has no solution then the line of three zeros will occur. The line of four zeros will occur if the system has infinite solutions. The calculator will not find the line or plane of solutions, but it will certainly make it easier.

Example

Find the general solution to this system of equations.

$$\begin{aligned} x + 3y + z &= 4 \\ 2x - y + 2z &= 3 \\ x - 4y + z &= -1 \end{aligned}$$

The augmented matrix for this system is $\left(\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & -1 & 2 & 3 \\ 1 & -4 & 1 & -1 \end{array}\right)$

The calculator display is shown below:

```
rref([A])
[[1 0 1 1.85714…
 [0 1 0 .714285…
 [0 0 0 0        …
```

The line of four zeros at the bottom indicates the infinite solutions. If we rewrite this in the form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.85714\ldots \\ 0.714285 \\ 0 \end{pmatrix}$$

we can see that $y = 0.714285$, but that x and z cannot be solved uniquely. Hence if we let $x = t$, then $t + z = 1.85714$

$$\Rightarrow z = -t + 1.85714$$

Hence the general solution to the equations is $x = t, y = 0.714285$ and $z = -t + 1.85714$. This is a case of three planes meeting in a line.

Exercise 4

1 Using elimination, solve these systems of equations.

a $6x + 8y + 5z = 1$, $3x + 5y + 3z = 3$, $2x + 3y + 2z = -1$

b $4x + 7y + 3z = 2$, $2x + 5y + 2z = -2$, $5x + 13y + 5z = 0$

c $x - 2y = 10$, $3x - y + z = 7$, $2x - y + z = 5$

d $4x + 8y + 3z = 6$, $3x + 5y + z = 3$, $4x + y + 4z = 15$

2 Using substitution, solve these systems of equations.

a $2x + 2y - z = -6$, $3x + 7y + 2z = 13$, $2x + 5y + 2z = 12$

b $2x + y - 2z = -11$, $x - 3y + 8z = 27$, $3x - 2y + z = -4$

c $3x + y + 4z = 6$, $2x - y - 2z = -7$, $x + 2y + 6z = 13$

d $2x + 3y + 4z = 1$, $4x - y - 6z = 9$, $x - 2y - 8z = -2$

3 Using a method of inverse matrices, solve these systems of equations.

a $x + 3y + 6z = 1$, $2x + 6y + 9z = 4$, $3x + 3y + 12z = 5$

b $x + 3y + 2z = 1$, $4x - y - 6z = 12$, $2x + y - 5z = 10$

c $2x - y - 2z = 4$, $4x + y - 3z = 9$, $6x - 2y - 3z = 7$

d $3x - y + z = 1$, $2x + 3y + 5z = 3$, $x - 2y - 5z = 6$

4 By evaluating the determinant, state whether each system of equations has a unique solution or not.

a $x + 2y - 5z = 15$, $2x - 3y + 7z = -1$, $3x + y - 2z = 12$

b $2x - y + 3z = 12$, $x - y + 7z = 15$, $3x - y - z = 7$

c $x + y - z = 4$, $2x + 2y - 3z = 4$, $3x + 3y - 2z = 8$

d $2x + y - 2z = 4$, $4x + 2y - 4z = 8$, $x + y - z = 2$

5 Using row operations, solve these systems of equations.

a $3x + 4y + 7z = 0$, $2x - y + 4z = 3$, $x + 2y + 5z = 2$

b $2x + y + 3z = 1$, $3x - 4y - 2z = 9$, $x - y - 2z = 0$

c $x - 3y + 2z = 13$, $2x - y + 2z = 8$, $3x + 3y + 2z = -1$

d $2x - y + 3z = 6$, $6x - y + z = 3$, $10x + 3y - 2z = 0$

6 Using a calculator, solve these systems of equations.

a $x + 2y - 3z = 5$, $2x - y + 2z = 7$, $3x + 2y + 5z = 9$

b $3x - 3y + z = 8$, $2x + y - 2z = 5$, $3x + 4y + z = 1$

c $3x + 3y - 6z = 2$, $6x - 8y + z = 8$, $x - y + 3z = 0$

d $3x + 2y - z = 1$, $6x - y + 3z = 7$, $11x + y + 2z = 8$

7 Without using a calculator, solve the following equations where possible.

a $x + 3y + 2z = 1$, $x + y - 2z = 4$, $x + 7y + 10z = -5$

b $2x + y + 3z = 4$, $-x + 2y + z = 2$, $x + 3y + 4z = 6$

c $3x - y + 4z = 1$, $x + 2y - 3z = 4$, $x - 5y + 10z = -7$

d $2x + y + 3z = 4$, $4x + 2y + 6z = 8$, $6x + 3y + 9z = 12$

e $x + 2y - z = 1$, $2x - y + 3z = 4$, $5x - 5y + 5z = 9$

f $-x - y + z = 4$, $2x - y + z = 8$, $y - z = 3$

g $3x + y - 2z = 4$, $x + y + z = 4$, $x + 3y + 6z = 10$

h $3x + y - 2z = 4$, $2x - y + z = 10$, $x - 4y + 4z = 12$

8 Using a calculator, state whether the following equations have a unique solution, no solution or infinite solutions. If the solution is unique, state it, and if the solution is infinite, give it in terms of one parameter.

a $2x + y - 2z = 4$, $x + 3y - 2z = 7$, $x + 8y - 4z = 17$

b $3x + 5y - 2z = 2$, $x + 7y + 4z = 1$, $3x - 3y + 2z = -2$

c $x + 4y - z = 4$, $-x + 7y = 10$, $3x + y - 2z = 7$

d $3x + y - 2z = 4$, $x + y + z = 4$, $x + 3y + 6z = 10$

e $x - y + 5z = 4$, $2x + y - z = 8$, $3y - 11z = 0$

f $x + y - z = 6$, $2x + y - z = 7$, $x + y - 5z = 18$

9 a Find the inverse of $\begin{pmatrix} 1 & -3 & -3 \\ 2 & -1 & 3 \\ 3 & -9 & 9 \end{pmatrix}$.

b Hence solve this system of equations.

$$x - 3y - 3z = 2$$
$$2x - y + 3z = 1$$
$$3x - 9y + 9z = 4$$

10 a Find the determinant of the matrix $\begin{pmatrix} 1 & 9 & 5 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

b Find the value of c for which this system of equations can be solved.

$$\begin{pmatrix} 1 & 4 & 5 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ c \\ 2 \end{pmatrix}$$

c Using this value of c, give the general solution to the system of equations.

11 Without finding the solution, show that this system of equations has a unique solution.

$$\begin{aligned} 2x - 5y + z &= 4 \\ x - y + 3z &= -4 \\ 4x + 4y + 3z &= 1 \end{aligned}$$

12 If the following system of equations does not have a unique solution, state the relationship between a and b.

$$\begin{aligned} ax - 3y + 2z &= 4 \\ 2x - 5y + z &= 9 \\ 2x - by + 4z &= -1 \end{aligned}$$

13 a Let $M = \begin{pmatrix} 1 & k & -3 \\ 2 & 4 & -5 \\ 3 & -1 & k \end{pmatrix}$. Find Det M.

b Find the value of k for which this system of equations does not have a unique solution.

$$\begin{aligned} x + ky - 3z &= 1 \\ 2x + 4y - 5z &= 2 \\ 3x - y + kz &= 3 \end{aligned}$$

14 Find the value of a for which this system of equations is consistent.

$$\begin{aligned} x - 3y + z &= 3 \\ x + 5y - 2z &= 1 \\ 16x - 2z &= a \end{aligned}$$

Review exercise

1 If P is an $m \times n$ matrix and Q is an $n \times p$ matrix find the orders of the matrices R and S such that $3P(-4Q + 2R) = 5S$.

2 Given that $A = \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find the values of λ for which $(A - \lambda I)$ is a singular matrix. [IB May 03 P1 Q5]

3 a Find the inverse of the matrix $A = \begin{pmatrix} k & -1 \\ 1 & k \end{pmatrix}$, where $k \in \mathbb{R}$.

b Hence or otherwise, solve the simultaneous equations

$$\begin{aligned} kx - y &= 2k \\ x + ky &= 1 - k^2 \end{aligned}$$

[IB Nov 97 P1 Q12]

4 If $A = \begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix}$, find the values of x and y, given that $AB = BA$. [IB Nov 01 P1 Q6]

5 Consider this system of equations.

$$\begin{aligned} x + (k + 3)y + 5z &= 0 \\ x + 3y + (k + 1)z &= k + 2 \\ x + y + kz &= 2k - 1 \end{aligned}$$

a Write the system in matrix form $AX = B$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

b Find the value of k for which the determinant of A is zero.

c Find the value of z in terms of k.

d Describe the solutions to the system of equations.

6 Given the following two matrices, $M = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & -2 \\ 1 & 2 & a \end{pmatrix}$ and

$M^{-1} = \frac{1}{2}\begin{pmatrix} b & -5 & 4 \\ -1 & 1 & 0 \\ 1 & -3 & 2 \end{pmatrix}$, find the values of a and b. [IB Nov 98 P1 Q14]

7 a Find the relationship between p, q and r such that the following system of equations has a solution.

$$\begin{aligned} 2x - y - 3z &= p \\ 3x + y + 4z &= q \\ -3x - 6y - 21z &= r \end{aligned}$$

b If $p = 3$ and $q = -1$, find the solution to the system of equations. Is this solution unique?

8 Let $A = \begin{pmatrix} 2 & 6 \\ k & -1 \end{pmatrix}$ and $B = \begin{pmatrix} h & 3 \\ -3 & 7 \end{pmatrix}$, where h and k are integers. Given that Det A = Det B and that Det $AB = 256h$,

a show that h satisfies the equation $49h^2 - 130h + 81 = 0$

b hence find the value of k. [IB May 06 P1 Q17]

9 If $M = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M^2 = pM + qI$, find the values of p and q.

10 a Find the values of c for which $M = \begin{pmatrix} c^3 & 3 & 8 \\ c & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ is singular.

b Find A where $A = \begin{pmatrix} 8 & 3 & 8 \\ 2 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 1 \end{pmatrix}$.

c Explain why A is singular.

11 The system of equations represented by the following matrix equation has an infinite number of solutions.

$$\begin{pmatrix} 2 & -1 & -9 \\ 1 & 2 & 3 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ k \end{pmatrix}$$

Find the value of k. [IB May 00 P1 Q6]

12 The variables x, y and z satisfy the simultaneous equations

$$\begin{aligned} x - 2y + z &= 3 \\ 2x + 3y + 4z &= 5 \\ -x - 2y - z &= c \end{aligned}$$

where c is a constant.

a Show that these equations do not have a unique solution.

b Find the value of c that makes these equations consistent.

c For this value of c, find the general solution to these equations.

13 a Find the values of a and b given that the matrix $A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$.

b For the values of a and b found in part **a**, solve the system of linear equations

$$\begin{aligned} x + 2y - 2z &= 5 \\ 3x + by + z &= 0 \\ -x + y - 3z &= a - 1 \end{aligned}$$

[IB Nov 99 P1 Q12]

14 Show that the following system of equations has a solution only when $p - 2q + r = 0$.

$$\begin{aligned} 3a - 5b + c &= p \\ 2a + b - 4c &= q \\ -a + 7b - 9c &= r \end{aligned}$$

15 Find the value of a for which the following system of equations does not have a unique solution.

$$\begin{aligned} 4x - y + 2z &= 1 \\ 2x + 3y &= -6 \\ x - 2y + az &= \frac{7}{2} \end{aligned}$$

[IB May 99 P1 Q6]

16 Given that $P = \begin{pmatrix} 2 & -1 & 3 \\ a & 2 & b \\ 4 & 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -2 & 1 \\ 3 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and

$R = \begin{pmatrix} -3 & -1 & -1 \\ 6 & -9 & 5 \\ 0 & -8 & 4 \end{pmatrix}$, find the values of a and b such that $PQ = R$.

17 Given the two sets of equations,

$$\begin{aligned} x_1 &= 3z_1 - 2z_2 + 5z_3 & \qquad y_1 &= x_1 + 4x_2 - 3x_3 \\ x_2 &= 4z_1 + 5z_2 - 9z_3 & \qquad y_2 &= 3x_1 - 5x_2 - 7x_3 \\ x_3 &= z_1 - 6z_2 + 9z_3 & \qquad y_3 &= 2x_1 + 2x_2 - x_3 \end{aligned}$$

use matrix methods to obtain three equations that express y_1, y_2 and y_3 directly in terms of z_1, z_2 and z_3.