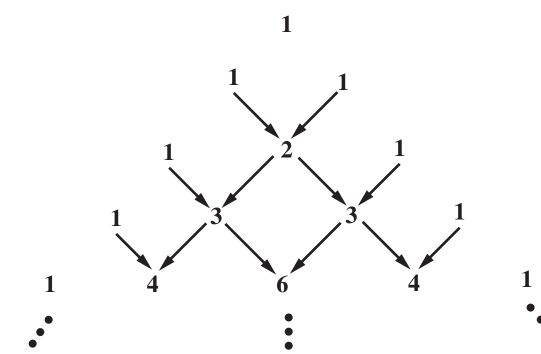
6 Sequences, Series and Binomial Theorem



Pascal's triangle is constructed by adding together the two numbers above as shown (with a 1 on the end of each row). There are many interesting results and applications related to this triangle. For example, notice that the sum of each row is a power of 2.

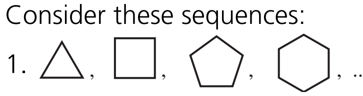
Pascal's triangle is named after Blaise Pascal, born 1623, a French mathematician who made great contributions to the fields of number theory, geometry and probability. However, it is not universally known as Pascal's triangle as it was not discovered first by him. There is evidence that Chinese and Persian mathematicians independently found the triangle as early as the 11th century. Chia Hsien, Yang Hui and Omar Khayyam are all documented as using the triangle. In fact, there may be reference to the triangle as early as 450 BC by an Indian mathematician who described the "Staircase of Mount Meru". In China the triangle is known as the Chinese triangle, and in Italy it is known as Tartaglia's triangle, named after a 16th century Italian mathematician, Nicolo Tartaglia.

http://www.bath.ac.uk/~ma3mja/history.html

Accessed 14 February 2006

A sequence is defined as an ordered set of objects. In most cases these objects are numbers, but this is not necessarily the case. Sequences and series occur in nature, such as the patterns on snail shells and seed heads in flowers, and in man-made applications such as the world of finance and so are a useful area of study. Whereas a sequence is a list of objects in a definite order, a series is the sum of these objects.

Consider these sequences:

1. 
2. J, A, S, O, N, D, J, …
3. M, W, F, S, T, T, S, …
4. Moscow, Los Angeles, Seoul, Barcelona, Atlanta, Sydney, Athens, Beijing, London, …
5. 2, 4, 6, 8, 10, …
6. 10, 13, 16, 19, 22, …
7. 3, 6, 12, 24, 48, …
8. 1, 3, 7, 15, 31, …

How can these sequences be described?

Here are some possible descriptions:

1. Plane shapes beginning with triangle, with one vertex (and side) added each time.
2. Initial letter of each month (in English) beginning with July.
3. Initial letter of days, starting with Monday, going forward each time by two days.
4. Olympic cities beginning with Moscow (1980).
5. Even numbers beginning with 2, increasing by 2 each time.
6. Numbers beginning with 10, increasing by 3 each time.
7. Beginning with 3, each term is the previous term multiplied by 2.
8. Beginning with 1, each term is double the previous term plus 1.

It is natural to describe a sequence by the change occurring each time from one term to the next, along with the starting point. Although all of the above are sequences, in this course only the types of which 5, 6 and 7 are examples are studied.

In order to describe sequences mathematically, some notation is required.

> u_n is known as the *n*th term of a sequence.
> This provides a formula for the general term of a sequence related to its term number, *n*.
> S_n is the notation for the sum of the first *n* terms.
> *a* or u_1 is commonly used to denote the initial term of a sequence.

In this course, two types of sequence are considered: arithmetic sequences and geometric sequences.

6.1 Arithmetic sequences

An arithmetic sequence, sometimes known as an arithmetic progression, is one where the terms are separated by the same amount each time. This is known as the **common difference** and is denoted by *d*. Note that for a sequence to be arithmetic, a common difference must exist.

Consider the sequence 5, 7, 9, 11, 13, …

The first term is 5 and the common difference is 2.
So we can say $a = 5$ and $d = 2$.

Sequences can be defined in two ways, explicitly or implicitly. An **implicit expression** gives the result in relation to the previous term, whereas an **explicit expression** gives the result in terms of *n*. Although it is very easy to express sequences implicitly, it is usually more useful to find an explicit expression in terms of *n*.

Here, an implicit expression could be $u_n = u_{n-1} + 2$.

*n*th term   previous term

For an explicit expression consider this table.

n	1	2	3	4	5
$\boldsymbol{u_n}$	5	7	9	11	13

In this case, $u_n = 2n + 3$.

> Compare this with finding the straight line with gradient 2 and *y*-intercept 3.

It is clear that an arithmetic sequence will be of the form

$a, a + d, a + 2d, a + 3d, a + 4d, \dots$

Hence, the general formula for the nth term of an arithmetic sequence is

$$u_n = a + (n - 1)d$$

Example

Consider the arithmetic sequence 4, 11, 18, 25, 32, …
(a) Find an expression for u_n, the nth term of the sequence.
(b) Find u_{12}, the 12th term of the sequence.
(c) Is (i) 602 (ii) 711 a member of this sequence?

(a) Clearly for this sequence $u_1 = a = 4$ and $d = 7$,
so $u_n = 4 + 7(n - 1)$
$u_n = 4 + 7n - 7$
$u_n = 7n - 3$

(b) Hence $u_{12} = 7 \times 12 - 3$
$= 81$

(c) (i) $7n - 3 = 602$
$\Rightarrow 7n = 605$
$\Rightarrow n = 86.4$
Since n is not an integer, 602 cannot be a term of this sequence.
(ii) $7n - 3 = 711$
$\Rightarrow 7n = 714$
$\Rightarrow n = 102$
Clearly 711 is a member of the sequence, the 102nd term.

Example

What is the nth term of a sequence with $u_7 = 79$ and $u_{12} = 64$?
If $u_7 = 79$, then $a + 6d = 79$.
If $u_{12} = 64$, then $a + 11d = 64$.
Subtracting, $-5d = 15$
$\Rightarrow d = -3$
Now substituting this into $a + 6d = 79$,
$a - 18 = 79$
$\Rightarrow a = 97$
Hence $u_n = 97 - 3(n - 1)$
$= 100 - 3n$
It is easy to verify that this is the correct formula by checking u_{12}.

Example

If k, 12, $k^2 - 6k$ are consecutive terms of an arithmetic sequence, find the possible values of k.
As the sequence is arithmetic, a common difference must exist.
Hence $d = 12 - k$ and $d = k^2 - 6k - 12$.
So $k^2 - 6k - 12 = 12 - k$
$\Rightarrow k^2 - 5k - 24 = 0$
$\Rightarrow (k + 3)(k - 8) = 0$
$\Rightarrow k = -3$ or $k = 8$

Exercise 1

1 Find u_n for these sequences.
a 5, 7, 9, 11, 13, …
b 1, 6, 11, 16, 21, …
c 8, 14, 20, 26, 32, …
d 60, 51, 42, 33, 24, …
e 4, 0, −4, −8, −12, …
2 For the sequence 7, 18, 29, 40, 51, … find u_n and u_{20}.
3 For the sequence 200, 310, 420, 530, 640, … find u_n and u_{13}.
4 For the sequence 17, 10, 3, −4, −11, … find u_n and u_{19}.
5 For the sequence $1, \frac{3}{2}, 2, \frac{5}{2}, 3, \dots$ find u_n and u_{15}.
6 For 9, 16, 23, 30, 37, … which term is the first to exceed 1000?
7 For 28, 50, 72, 94, 116, … which term is the first to exceed 500?
8 For 160, 154, 148, 142, 136, … which term is the last positive term?
9 Find u_n given $u_5 = 17$ and $u_9 = 33$.
10 Find u_n given $u_4 = 43$ and $u_{10} = 97$.
11 Find u_n given $u_3 = 32$ and $u_7 = 8$.
12 Find u_n given $u_8 = -8$ and $u_{14} = -11$.
13 Given that k, 8, $7k$ are consecutive terms of an arithmetic sequence, find k.
14 Given that $k - 1$, 11, $2k - 1$ are consecutive terms of an arithmetic sequence, find k.
15 Given that $4k - 2$, 18, $9k - 1$ are consecutive terms of an arithmetic sequence, find k.
16 Given that $k^2 + 4$, 29, $3k$ are consecutive terms of an arithmetic sequence, find k.

6.2 Sum of the first n terms of an arithmetic sequence

An arithmetic series is the sum of an arithmetic sequence.

So for $u_n = 3n + 5$, i.e. 8, 11, 14, 17, 20, … , the arithmetic series is
$S_n = 8 + 11 + 14 + 17 + 20 + \dots$

So S_5 means $8 + 11 + 14 + 17 + 20 = 70$.

How can a formula for S_n be found?

$S_n = u_1 + u_2 + \ldots + u_{n-1} + u_n = a + (a + d) + \ldots + [a + (n - 2)d] + [a + (n - 1)d]$

Re-ordering,

$S_n = u_n + u_{n-1} + \ldots + u_2 + u_1 = [a + (n - 1)d] + [a + (n - 2)d] + \ldots + (a + d) + a$

Adding,

$$2S_n = 2a + (n - 1)d + 2a + (n - 1)d + \ldots + 2a + (n - 1)d + 2a + (n - 1)d$$
$$= n[2a + (n - 1)d]$$
$$\Rightarrow S_n = \frac{n}{2}[2a + (n - 1)d]$$

This is the formula for S_n. It can be expressed in two ways:

$$S_n = \frac{n}{2}[2a + (n - 1)d] \qquad S_n = \frac{n}{2}[u_1 + u_n]$$

This is because
$u_n = a + (n - 1)d$

So in the above example, $S_n = \frac{n}{2}[16 + 3(n - 1)]$

$$= \frac{n}{2}(16 + 3n - 3)$$
$$= \frac{n}{2}(3n + 13)$$
$$= \frac{3}{2}n^2 + \frac{13}{2}n$$

Example

Find a formula for S_n for 7, 15, 23, 31, 39, … and hence find S_8.

Here $a = 7$ and $d = 8$.

So $S_n = \frac{n}{2}[14 + 8(n - 1)]$ and $S_8 = 4 \times 64 + 3 \times 8$

$S_n = \frac{n}{2}(8n + 6)$ $\qquad S_8 = 256 + 24$

$S_n = 4n^2 + 3n$ $\qquad S_8 = 280$

Example

Find the number of terms in the series $4 + 10 + 16 + 22 + 28 + \ldots$ required to exceed 500.

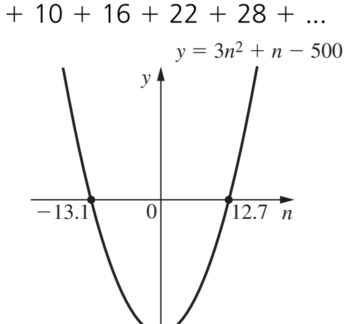
$a = 4, d = 6$

$S_n = \frac{n}{2}[8 + 6(n - 1)]$

$S_n = 3n^2 + n$

So $3n^2 + n > 500$

$\Rightarrow 3n^2 + n - 500 > 0$



So $n > 12.7$ (The solution of -13.1 is not valid as n must be positive.)

The number of terms required in the series is 13 (and $S_{13} = 520$).

Example

Given $S_4 = 32$ and $S_7 = 98$, find u_{11}.

$S_4 = \frac{4}{2}(2a + 3d) = 32$ $\qquad S_7 = \frac{7}{2}(2a + 6d) = 98$

$\Rightarrow 4a + 6d = 32$ $\qquad \Rightarrow 7a + 21d = 98$

$\Rightarrow 2a + 3d = 16$ (i) $\qquad \Rightarrow a + 3d = 14$ (ii)

$$2a + 3d = 16 \quad \text{(i)}$$
$$\text{(i)} - \text{(ii)} \quad \underline{a + 3d = 14} \quad \text{(ii)}$$
$$a = 2$$

Substituting in (i)

$a + 3d = 14$

$\Rightarrow 3d = 12$

$\Rightarrow d = 4$

So $u_n = a + (n - 1)d = 2 + 4(n - 1)$

$u_n = 4n - 2$

$\Rightarrow u_{11} = 44 - 2 = 42$

Exercise 2

1 Find a formula for S_n for these series.

a $2 + 5 + 8 + 11 + 14 + \ldots$

b $8 + 10 + 12 + 14 + 16 + \ldots$

c $80 + 77 + 74 + 71 + 68 + \ldots$

d $2008 + 1996 + 1984 + 1972 + 1960 + \ldots$

e $\frac{1}{2} + \frac{5}{6} + \frac{7}{6} + \frac{3}{2} + \frac{11}{6} + \ldots$

2 Find S_7 for $8 + 15 + 22 + 29 + 36 + \ldots$.

3 For the series obtained from the arithmetic sequence $u_n = 5n - 3$, find S_{12}.

4 Find the sum of the first 20 multiples of 5 (including 5 itself).

5 Find the sum of the multiples of 7 between 100 and 300.

6 Find an expression for the sum of the first n positive integers.

7 Find an expression for the sum of the first n odd numbers.

8 Given that three consecutive terms of an arithmetic sequence add together to make 30 and have a product of 640, find the three terms.

9 Find the number of terms in the arithmetic series $9 + 14 + 19 + 24 + \ldots$ required to exceed 700.

10 Find the greatest possible number of terms in the arithmetic series $18 + 22 + 26 + \ldots$ such that the total is less than 200.

11 What is the greatest total possible (maximum value) of the arithmetic series $187 + 173 + 159 + \ldots$?

12 In an arithmetic progression, the 10th term is twice the 5th term and the 30th term of the sequence is 60.

a Find the common difference.

b Find the sum of the 9th to the 20th terms inclusive.

6.3 Geometric sequences and series

An example of a geometric sequence is 4, 8, 16, 32, 64, …

In a geometric sequence each term is the previous one multiplied by a non-zero constant. This constant is known as the **common ratio**, denoted by r.

The algebraic definition of this is:

$$\frac{u_{n+1}}{u_n} = r \Leftrightarrow \text{the sequence is geometric}$$

Formula for u_n

A geometric sequence has the form

$a, ar, ar^2, ar^3, \ldots$

$$\text{So } u_n = ar^{n-1}$$

Example

Find a formula for u_n for the geometric sequence
6, 12, 24, 48, 96, …

Here $a = 6$ and $r = 2$.
So $u_n = 6 \times 2^{n-1}$.

Example

Find u_n given that $u_3 = 36$ and $u_5 = 324$.

$u_3 = ar^2 = 36 \qquad u_5 = ar^4 = 324$

So $\frac{u_5}{u_3} = \frac{ar^4}{ar^2} = r^2 = \frac{324}{36}$

So $r^2 = 9$

$\Rightarrow r = \pm 3$

$ar^2 = 36$

$\Rightarrow 9a = 36$

$\Rightarrow a = 4$

i.e. $u_n = 4 \times 3^{n-1}$ or $u_n = 4 \times (-3)^{n-1}$

This technique of dividing one term by another is commonly used when solving problems related to geometric sequences.

Example

Given that the following are three consecutive terms of a geometric sequence, find k.

$k - 4, 2k - 2, k^2 + 14$

So $r = \frac{2k-2}{k-4} = \frac{k^2+14}{2k-2}$

$\Rightarrow (2k-2)^2 = (k-4)(k^2+14)$

$\Rightarrow 4k^2 - 8k + 4 = k^3 + 14k - 4k^2 - 56$

$\Rightarrow k^3 - 8k^2 + 22k - 60 = 0$

$\Rightarrow k = 6$

Use a calculator to solve the cubic equation.

Sum of a geometric series

As with arithmetic series, a geometric series is the sum of a geometric sequence.

So $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

$\Rightarrow rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$\Rightarrow rS_n = S_n + ar^n - a$

$\Rightarrow rS_n - S_n = ar^n - a$

$\Rightarrow S_n(r-1) = a(r^n - 1)$

$\Rightarrow S_n = \frac{a(r^n-1)}{r-1}$

The formula for S_n can be expressed in two ways:

$$S_n = \frac{a(r^n-1)}{r-1} \quad \text{or} \quad S_n = \frac{a(1-r^n)}{1-r}$$

Example

Find S_n for $4 - 8 + 16 - 32 + 64$.
Here $a = 4$ and $r = -2$.

So $S_n = \frac{4((-2)^n - 1)}{-2-1}$

$= -\frac{4}{3}((-2)^n - 1)$

Exercise 3

1 Find the 6th term and the nth term for these geometric sequences.
- **a** 8, 4, 2, …
- **b** 80, 20, 5, …
- **c** 2, 6, 18, …
- **d** 5, −10, 20, …
- **e** 100, −50, 25, …
- **f** $u_1 = 12, r = 2$
- **g** $a = 6, r = 5$

2 Find the sum of the first eight terms for each of the sequences in question 1. Also find the sum to n terms of these numerical sequences.

3 Find the sum to n terms of these geometric sequences.

a $x + x^2 + x^3 + \cdots$

b $1 - x + x^2 - \cdots$

c $1 - 3x + 9x^2 - 27x^3 + \cdots$

4 Find the general term, u_n, of the geometric sequence that has:

a $u_3 = 20$ and $u_6 = 160$

b $u_2 = 90$ and $u_5 = \frac{10}{3}$

c $u_2 = -12$ and $u_5 = 324$

d $u_2 = -\frac{1}{2}$ and $u_7 = 512$

5 Given these three consecutive terms of a geometric sequence, find k.

a $k - 4, k + 8, 5k + 4$

b $k - 1, 2 - 2k, k^2 - 1$

c $\frac{k}{2}, k + 8, k^2$

6 Find the first term in this geometric sequence that exceeds 500.
$2, 4, 8, 16, \ldots$

7 If $a = 8$ and $r = 4$, find the last term that is less than 8000.

8 For the geometric series $3 + 6 + 12 + 24 + \cdots$, how many terms are required for a total exceeding 600?

9 The first two terms of a geometric series have a sum of -4. The fourth and fifth terms have a sum of 256. Find the first term and the common ratio of the series.

6.4 Sum of an infinite series

In order to consider infinite series, it is first important to understand the ideas of convergence and divergence. If two (or more) things converge, then they move towards each other. In a sequence or series, this means that successive terms become closer and closer together; to test convergence, the gap between the terms is examined.

Consider these three series:

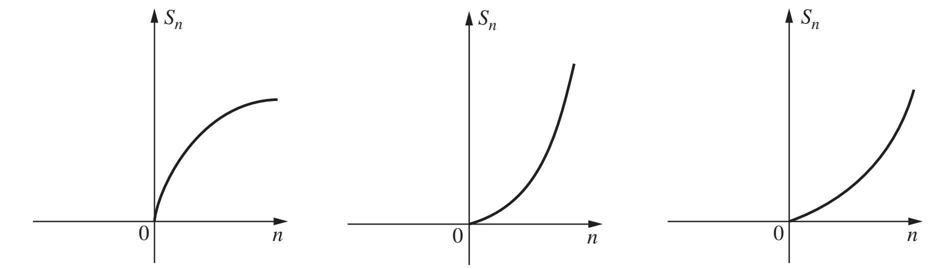
1. $24 + 12 + 6 + 3 + \cdots$
2. $5 + 10 + 20 + 40 + \cdots$
3. $4 + 7 + 10 + 13 + \cdots$

In the first (geometric) series, the gap between the terms narrows so the series is said to be **convergent**.

In the second (geometric) series, the gap between the terms widens and will continue to increase, so the series is said to be **divergent**.

The third series is arithmetic and so the gap between the terms remains the same throughout, known as the common difference. Although the gap remains constant, the series continues to increase in absolute size towards infinity and hence all arithmetic series are divergent.

Plotting a graph of the above series can help to visualize what is happening with these series.



All of the above series are infinite but only the first series converges. Finding the infinite sum of a divergent series does not make any sense, and hence in order to find the sum to infinity of a series, the series must converge.

In order to find a result for the sum of an infinite series, it is important to understand the concept of a **limit**.

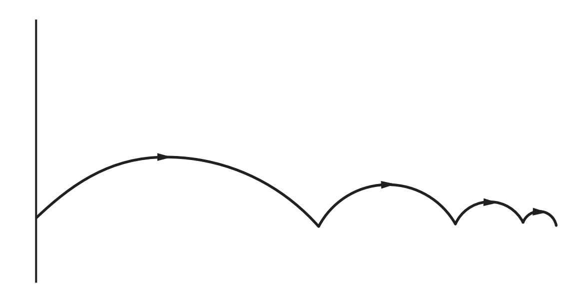
The concept of a limit is not particularly easy to define. The formal definition can be stated as

"A number or point L that is approached by a function $f(x)$ as x approaches a if, for every positive number ε, there exists a number δ such that $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$."

This is not necessarily helpful in visualizing the meaning of the term. A more informal viewpoint may help.

Consider Freddie Frog, who gets tired very quickly. Freddie hops 2 metres on his first hop. On his second hop, he is tired and can hop only half the distance, 1 metre. This continues, and each time he can hop only half the distance of his previous hop.

Consider Freddie trying to hop across a 4 metre road:



With each hop, he gets closer to the other side, but will he ever make it across the road? The distance that he has hopped can be considered to be

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

It is clear that he is getting very close to a distance of 4 metres but, as each hop is only half of his previous hop (and therefore half of the remaining distance), he will never actually reach 4 metres. In this situation, 4 metres is considered to be the limit of the distance hopped.

A limit is a value that a function or series approaches and becomes infinitesimally close to but will never reach. This idea has been covered in Chapter 3 – a horizontal asymptote is a value that a function approaches as x becomes large but never reaches. It is said that a series converges to a limit.

The notation for this is $\lim_{n \to \infty} S_n = L$ where L is the limit.

The formal definition for a limit in relation to functions given above is also true for series.

$\lim_{n \to \infty} S_n = L$ is true provided that S_n can be made as close to L as required by choosing n sufficiently large. In mathematical notation this can be stated "Given any number $\varepsilon > 0$, there exists an integer N such that $|S_n - L| < \varepsilon$ for all $n \geq N$".

Returning to the consideration of geometric series, will all infinite series converge to a limit? It is clear that the above series describing the frog does converge to a limit. However, consider the series

$1 + 10 + 100 + 1000 + 10000 + \ldots$

It is immediately clear that this series will continue to grow, and the gap between terms will continue to grow.

This is the key to understanding whether a series will converge – the gap between successive terms. If this gap is decreasing with each term, then the series will ultimately converge. Hence for the sum of a geometric series to converge, the common ratio must be reducing the terms. Putting this into mathematical notation,

a series will only converge if $|r| < 1$

If $|r| < 1$, then r^n will have a limit of zero as n becomes very large.

Considering the formula for the sum of n terms of a geometric series,

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a - ar^n}{1 - r}$$

For large values of n (as n approaches ∞), $ar^n \to 0$ if $|r| < 1$.

Think of 0.1^{10}, 0.1^{50} etc.

So when $n \to \infty$, the sum becomes

$$S_\infty = \frac{a}{1 - r} \quad (|r| < 1)$$

This formula can be used to find the limit of a convergent series, also known as the **sum to infinity** or infinite sum.

Example

Show that the sum to infinity of $8 + 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ exists and find this sum.

Clearly this series converges as $|r| = \frac{1}{4} < 1$.

So $S_\infty = \dfrac{a}{1 - r} = \dfrac{8}{\frac{3}{4}} = \dfrac{32}{3}$.

A recurring decimal such as $5.\dot{8} = 5.8888888\ldots$ can be considered to be an infinite geometric series as it is $5 + \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \ldots$. This means that the formula for the sum to infinity can be used to find an exact (fractional) value for the decimal. This is demonstrated by example.

Example

Find the exact value of the recurring decimal $1.\dot{2}$.

This can be considered as $1 + \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \ldots$

So the decimal part is a geometric series with $a = \frac{2}{10}$ and $r = \frac{1}{10}$.

Hence a limit exists since $r = \frac{1}{10} < 1$.

$$S_\infty = \frac{a}{1 - r} = \frac{\frac{2}{10}}{\frac{9}{10}} = \frac{2}{9}.$$

So we can write $1.\dot{2} = 1 + \frac{2}{9} = \frac{11}{9}$.

Exercise 4

Determine whether the series below converge. If they do, find the sum to infinity.

1 $20 + 10 + 5 + \cdots$

2 $81 + 27 + 9 + \cdots$

3 $4 + 12 + 36 + \cdots$

4 $-64 + 40 - 25 + \cdots$

5 $8 - 12 + 18 - \cdots$

Find the sum to infinity for the geometric series with:

6 $a = 6, r = \frac{1}{2}$

7 $a = 100, r = \frac{2}{3}$

8 $a = 60, r = -\frac{1}{5}$

9 $a = 9, r = -\frac{3}{4}$

Find the range of values of x for which the following series converge.

10 $1 + x + x^2 + x^3 + \ldots$

11 $4x - 4 + \frac{4}{x} - \frac{4}{x^2} + \ldots$

Find the exact value of these recurring decimals.

12 $6.\dot{4}$

13 $2.\dot{1}\dot{6}$

14 $7.\dot{3}\dot{4}$

15 Find the sum

a of the even numbers from 50 to 100 inclusive

b of the first ten terms of the geometric series that has a first term of 16 and a common ratio of 1.5

c to infinity of the geometric series whose second term is $\frac{2}{3}$ and third term $\frac{1}{2}$.

6.5 Applications of sequences and series

Although sequences and series occur naturally and in many applications, these mostly involve more complicated series than met in this course. Most common examples of geometric series at this level model financial applications and population.

Example

Katherine receives €200 for her twelfth birthday and opens a bank account that provides 5% compound interest per annum (per year). Assuming she makes no withdrawals nor any further deposits, how much money will she have on her eighteenth birthday?

This can be considered as a geometric series with $a = 200$ and $r = 1.05$.
The common ratio is 1.05 because 5% is being added to 100%, which gives $105\% = 1.05$.

So in six years the balance will be $u_7 = 200 \times 1.05^6$

$\Rightarrow u_7 = 268.02$

So she will have €268.02 on her eighteenth birthday.

If Katherine receives €200 on every birthday following her twelfth, how much will she have by her eighteenth?

After one year the balance will be $u_2 = 1.05 \times 200 + 200$.

After two years, the balance will be $u_3 = 1.05u_1 + 200$.

This can be expressed as $u_3 = 1.05(1.05 \times 200 + 200) + 200$

$= 1.05^2 \times 200 + 1.05 \times 200 + 200$

$= 200(1.05^2 + 1.05 + 1)$

So $u_7 = 200(1.05^6 + 1.05^5 + \ldots + 1.05 + 1)$

The part in brackets is a geometric series with $a = 1$ and $r = 1.05$.

So the sum in brackets is $\dfrac{1(1 - 1.05^6)}{1 - 1.05}$

$= 6.8019\ldots$

Hence the balance on her eighteenth birthday will be
$200 \times 6.8019\ldots = €1360.38$.

u_1 is the first term. After six years the balance will be u_7.

Exercise 5

1 In his training, Marcin does 10 sit-ups one day, then 12 sit-ups the following day. If he continues to do 2 more each day, how long before he completes 1000 sit-ups?

2 Karen invests \$2000 in an account paying 8% per year. How much will be in the account after 4 years?

3 Anders invests 50 000 DKr (Danish kroner) at 12% per year. How much is it worth after 6 years?

4 A kind benefactor sets up a prize in an international school. The benefactor donates £10 000. The school invests the money in an account paying 5% interest. If £750 is paid out annually, for how long can the full prize be given out?

5 If Yu wants to invest 50 000 yen with a return of 20 000 yen over 8 years, what % rate must she find?

6 What initial investment is required to produce a final balance of £12 000 if invested at 8% per year over 4 years?

7 In a Parisian sewer, the population of rats increases by 12% each month.

a If the initial population is 10 000, how many rats will there be after 5 months?

b How long before there are 50 000 rats?

8 The number of leopards in a Kenyan national park has been decreasing in recent years. There were 300 leopards in 2000 and the population has decreased at a rate of 9% annually.

a What was the population in 2005?

b When will the population drop below 100?

9 Each time a ball bounces, it reaches 85% of the height reached on the previous bounce. It is dropped from a height of 5 metres.

a What height does the ball reach after its third bounce?

b How many times does it bounce before the ball can no longer reach a height of 1 metre?

6.6 Sigma notation

Sigma is the Greek letter that corresponds to S in the Roman alphabet, and is written σ or Σ. The σ form is often used in statistics but the capital form Σ is used to denote a sum of discrete elements. This notation is a useful shorthand rather than writing out a long string of numbers. It is normally used on the set of integers.

Consider $1 + 2 + 3 + 4 + 5 + 6 + 7$

This can be written as $\sum_{r=1}^{7} r$

Last element $(r = 7)$ (ending at 7)

This is the rth term of the sequence.

First element $(r = 1)$ (starting from 1)

Similarly,

$$2 + 5 + 8 + 11 + 14 = \sum_{r=1}^{5} 3r - 1$$

We know that $3r - 1$ is the rth term because when $r = 1, 3r - 1 = 2$ and when $r = 2, 3r - 1 = 5$, etc.

Both arithmetic and geometric series can be expressed using this notation.

Example

Consider the arithmetic series

$100 + 96 + 92 + \ldots + 60$

Express the series using sigma notation.

This has $a = 100$ and $d = -4$. So the corresponding sequence has the general term

$$\begin{aligned} u_n &= 100 - 4(n - 1) \\ &= 104 - 4n \end{aligned}$$

This series can be expressed as $\sum_{r=1}^{11} 104 - 4r$.

The first n terms (i.e. S_n) could be expressed as $\sum_{r=1}^{n} 104 - 4r$.

Example

Consider the geometric series

$4 + 8 + 16 + 32 + \ldots + n$

Express the series using sigma notation.

This can be expressed as $\sum_{r=1}^{n} 4 \times 2^{r-1}$ or $\sum_{r=1}^{n} 2^{r+1}$.

Example

Express the sum $16 + 4 + 1 + \frac{1}{4} + \ldots$ using sigma notation.

The infinite sum $16 + 4 + 1 + \frac{1}{4} + \ldots$ can be expressed as $\sum_{r=1}^{\infty} 16 \times \left(\frac{1}{4}\right)^{r-1}$

There are results that we can use with sigma notation that help to simplify expressions. These are presented here without proof but are proved in Chapter 18.

Result 1

$$\sum_{r=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

$$\sum_{r=1}^{n} a = an \text{ where } a \text{ is a constant.}$$

This should be obvious as $\sum_{r=1}^{n} a = a + a + a + \cdots + a = na$

We can also see that a constant can be removed outside a sum:

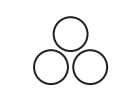
$$\sum_{r=1}^{n} a = a \sum_{r=1}^{n} 1$$

Result 2

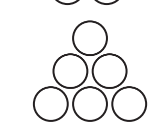
$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

This is the sum of the first n natural numbers, and each of these sums is also a triangular number (because that number of objects can be arranged as a triangle).

For example, $\sum_{r=1}^{2} r = \frac{2(2+1)}{2} = 3$



$\sum_{r=1}^{3} r = \frac{3(3+1)}{2} = 6$



Result 3

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

These three results can be used to simplify other sigma notation sums. Note that they apply only to sums beginning with $r = 1$. If the sums begin with another value the question becomes more complicated, and these are not dealt with in this curriculum.

Example

Simplify $\sum_{r=1}^{n} 4r^2 - 3r - 5$ and hence find $\sum_{r=1}^{6} 4r^2 - 3r - 5$.

$$\begin{aligned} \sum_{r=1}^{n} 4r^2 - 3r - 5 &= 4\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r - 5\sum_{r=1}^{n} 1 \\ &= 4 \times \frac{1}{6}n(n+1)(2n+1) - 3\frac{n(n+1)}{2} - 5n \end{aligned}$$

$= \frac{1}{6}n[(4n + 4)(2n + 1) - 9(n + 1) - 30]$

$= \frac{1}{6}n(8n^2 + 12n + 4 - 9n - 9 - 30)$

$= \frac{1}{6}n(8n^2 + 3n - 35)$

Hence $\sum_{r=1}^{6} 4r^2 - 3r - 5 = \frac{1}{6} \times 6(8 \times 6^2 + 3 \times 6 - 35)$

$= 288 + 18 - 35$

$= 271$

Exercise 6

1 Evaluate

a $\sum_{r=1}^{5} 3r - 2$ **b** $\sum_{i=4}^{7} 2i^2$ **c** $\sum_{k=3}^{8} 5k^2 - 3k$

2 Express each of these sums in sigma notation.

a $4 + 8 + 12 + 16 + 20$

b $-2 + 3 + 8 + 13 + \cdots + (5n + 3)$

c $9 + 13 + 17 + 21 + \cdots$

3 Use the results for $\sum_{r=1}^{n} 1$, $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to simplify these.

a $\sum_{r=1}^{n} 6r - 2$ **b** $\sum_{k=1}^{n} 2k^2 - k + 3$ **c** $\sum_{k=1}^{2n} 9 - k^2$ **d** $\sum_{r=1}^{k+1} 7r - 3$

6.7 Factorial notation

Sigma notation is a method used to simplify and shorten sums of numbers. There are also ways to shorten multiplication, one of which is factorial notation.

A factorial is denoted with an exclamation mark ! and means the product of all the positive integers up to that number.

$n! = 1 \times 2 \times \cdots \times (n - 1) \times n$

So $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

It is important to be able to perform arithmetic with factorials, as demonstrated in the examples below.

It is worth noting that 0! is defined to be 1.

Example

Simplify $\frac{8!}{5!}$.

It should be obvious that $8! = 8 \times 7 \times 6 \times 5!$ so $\frac{8!}{5!}$ can be simplified to $\frac{8 \times 7 \times 6 \times 5!}{5!} = 8 \times 7 \times 6 = 336$

Example

Simplify $\frac{n!}{(n-2)!}$

Using the same rationale as above,

$\frac{n!}{(n-2)!} = \frac{n \times (n-1) \times (n-2)!}{(n-2)!} = n(n-1)$

Example

Simplify $n! - (n - 2)!$

This can be factorised with a common factor of $(n - 2)!$

So $n! - (n - 2)! = (n - 2)!\,[n(n - 1) - 1]$

$= (n - 2)!\,(n^2 - n - 1)$

Permutations and combinations

Factorial notation is used most commonly with counting methods known as permutations and combinations.

Factorials can be used to determine the number of ways of arranging n objects.

Consider four people standing in a line: Anna, Julio, Mehmet and Shobana. How many different orders can they stand in?

The different ways can be listed systematically:

A	J	M	S
A	J	S	M
A	M	J	S
A	M	S	J
A	S	M	J
A	S	J	M
J	A	M	S
J	A	S	M
J	M	A	S
J	M	S	A
J	S	A	M
J	S	M	A
M	A	J	S
M	A	S	J
M	J	A	S
M	J	S	A
M	S	A	J
M	S	J	A
S	A	J	M
S	A	M	J
S	J	A	M
S	J	M	A
S	M	A	J
S	M	J	A

There are clearly 24 possibilities. This comes as no surprise as this can be considered as having 4 ways of choosing position 1, then for each choice having 3 ways of choosing position 2, and for each choice 2 ways of choosing position 3, leaving only 1 choice for position 4 each time.

This is equivalent to having $4 \times 3 \times 2 \times 1$ possibilities.

So $n!$ is the number of ways of arranging n objects in order.

Consider a bag with five balls in it, labelled A, B, C, D and E.

If two balls are chosen from the bag at random, there are 10 possible arrangements:

A&B A&C A&D A&E B&C B&D B&E C&D C&E D&E

If the order that the balls come out in matters, there would be 20 possible outcomes.

AB AC AD AE BC BD BE CD CE DE

BA CA DA EA CB DB EB DC EC ED

The first type are known as **combinations** (where order does not matter) and the second type as **permutations** (where order is important).

The formula for the number of combinations when choosing r objects at random from n objects is $\frac{n!}{r!\,(n-r)!}$.

There are two notations for combinations, ${}^nC_r$ or $\binom{n}{r}$.

$$\text{So } \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

The formula for the number of permutations when choosing r objects at random from n objects is very similar:

$${}^nP_r = \frac{n!}{(n-r)!}$$

This makes it clear that the $r!$ in the combinations formula removes the duplication of combinations merely in a different order. This topic is further developed in Chapter 20 in its application to probability.

It is important that we recognize whether we are working with a permutation or a combination, i.e. does order matter?

Example

How many 5 letter words (arrangements of letters) can be made from the letters of EIGHTYFOUR?

Here, the order of the letters matters so the number of words will be given by ${}^{10}P_5$.

${}^{10}P_5 = \frac{10!}{5!} = 30240$

Example

How many different hockey teams (11 players) can be chosen from a squad of 15? Here, the order in which the players are chosen is unimportant.

So the number of different teams is ${}^{15}C_{11} = \frac{15!}{11!\,(15-11)!} = 1365$.

Many calculators have in-built formulae for permutations and combinations.

Pascal's triangle revisited

Here is Pascal's triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Notice that this could also be written as

Row 1 $\binom{1}{0}\binom{1}{1}$

Row 2 $\binom{2}{0}\binom{2}{1}\binom{2}{2}$

Row 3 $\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$

Row 4 $\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$ etc.

So Pascal's triangle is also given by the possible combinations in each row n. This leads to recognizing some important results about combinations.

Result 1

$$\binom{n}{0} = \binom{n}{n} = 1$$

This is fairly obvious from the definition of ${}^nC_r$.

$\binom{n}{0} = \frac{n!}{n!\,0!} = \frac{n!}{n!} = 1$ $\binom{n}{n} = \frac{n!}{n!\,0!} = \frac{n!}{n!} = 1$

Result 2

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$\binom{n}{1} = \dfrac{n!}{(n-1)!\,1!} = \dfrac{n!}{(n-1)!} = n \qquad \binom{n}{n-1} = \dfrac{n!}{(n-1)!\,1!} = \dfrac{n!}{(n-1)!} = n$

The above two results and the symmetry of Pascal's triangle lead to result 3.

Result 3

$$\binom{n}{r} = \binom{n}{n-r}$$

This is again easy to show:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{[n-(n-r)]!\,(n-r)!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{n-r}$$

Result 4

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

This is the equivalent statement of saying that to obtain the next row of Pascal's triangle, add the two numbers above.

The proof of result 4 is as follows.

$$\binom{n}{r-1} + \binom{n}{r}$$

$$= \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$$

$$= \frac{r \cdot n!}{r!\,(n-r+1)!} + \frac{(n-r+1)\cdot n!}{r!\,(n-r+1)!}$$

$$= \frac{r \cdot n! + (n+1)\cdot n! - r \cdot n!}{r!\,(n-r+1)!}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!}$$

$$= \binom{n+1}{r}$$

Example

Solve

$$\binom{n+1}{1} + \binom{n+1}{2} = 66$$

From result 4

$\Rightarrow \binom{n+2}{2} = 66$

$\Rightarrow \dfrac{(n+2)!}{(n+2-2)!\,2!} = 66$

$\Rightarrow \dfrac{(n+2)!}{n! \times 2} = 66$

$\Rightarrow (n+2)(n+1) = 132$

$\Rightarrow n^2 + 3n - 130 = 0$

$\Rightarrow (n+13)(n-10) = 0$

$\Rightarrow n = 10$

Remember that n must be positive.

Exercise 7

1 Evaluate the following:

a ${}^6P_2$ **b** ${}^8P_3$ **c** ${}^8C_3$ **d** $\binom{9}{5}$ **e** $\binom{8}{4}$

2 How many different 4 letter words (arrangements where order matters) can be made from the letters A, E, I, O, U, Y?

3 How many different committees of 9 can be made from 14 people?

4 A grade 5 class has 11 students.

a If the teacher lines them up, how many different orders can there be?

b If 3 students are selected as president, secretary and treasurer of the Eco-Club, how many different ways can this be done?

c If 7 students are chosen for a mini-rugby match, how many different teams are possible?

5 In the UK national lottery, 6 balls are chosen at random from 49 balls. In the Viking lottery operated in Scandinavia, 6 balls are chosen from 48 balls. How many more possible combinations result from the extra ball?

6 The EuroMillions game chooses 5 numbers at random from 50 balls and then 2 more balls known as lucky stars from balls numbered 1–9. How many possible combinations are there for the jackpot prize (5 numbers plus 2 lucky stars)?

7 José is choosing his 11 players for a soccer match. Of his squad of 20, one player is suspended. He has three players whom he always picks (certainties). How many possible teams can he create?

8 How many 3-digit numbers can be created from the digits 2, 3, 4, 5, 6 and 7 if each digit may be used

a any number of times **b** only once.

9 Solve these equations.

a $\binom{n}{2} = 15$ **b** $\binom{n}{3} = 10$ **c** $\binom{2n}{2} = 28$ **d** $\binom{2n}{2} = 66$

10 Solve these equations.

a $\binom{n}{n-2} = 6$ **b** $\binom{n}{n-2} = 45$ **c** $\binom{n}{n-3} = 84$

11 Find a value of n that satisfies each equation.

a $\binom{n}{1} + \binom{n}{2} = 28$ **b** $\binom{n+2}{2} + \binom{n+2}{3} = 20$ **c** $\binom{2n}{3} + \binom{2n}{4} = 35$

6.8 Binomial theorem

The binomial theorem is a result that provides the expansion of $(x + y)^n$.
Consider the expansions of $(x + y)^1$, $(x + y)^2$ and $(x + y)^3$.

$(x + y)^1 = 1x + 1y$ $\quad (x + y)^2 = 1x^2 + 2xy + 1y^2$ $\quad (x + y)^3 = (x + y)(x^2 + 2xy + y^2)$

$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

Notice that the coefficients are the same as the numbers in Pascal's triangle.

Similarly, $(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

From Pascal's triangle, this could be rewritten

$$(x + y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

This leads to the general expansion

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

This can be shortened to

$$(x + y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

This result is stated here without proof; the proof is presented in Chapter 18.

A useful special case is the expansion of $(1 + x)^n$

$$(1 + x)^n = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

Example

Using the binomial theorem, expand $(2x + 3)^5$.

This can be written $(2x + 3)^5 = \sum_{r=0}^{5}\binom{5}{r}(2x)^{5-r}3^r$

So

$$(2x+3)^5 = \binom{5}{0}2^5x^53^0 + \binom{5}{1}2^4x^43^1 + \binom{5}{2}2^3x^33^2 + \binom{5}{3}2^2x^23^3 + \binom{5}{4}2^1x^13^4 + \binom{5}{5}2^0x^03^5$$

$$= 32x^5 + 5 \times 16 \times 3x^4 + 10 \times 8 \times 9x^3 + 10 \times 4 \times 27x^2$$
$$+ 5 \times 2 \times 81x + 243$$
$$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$$

Example

Using the result for $(1 + x)^n$, find the expansion of $(4 + 3x)^4$.

$(4 + 3x)^4 = 4\left(1 + \frac{3}{4}x\right)^4$

Using the result for $(1 + x)^n$, this becomes

$$4\left[1 + 4\left(\frac{3}{4}x\right) + \frac{4 \times 3}{2!}\left(\frac{3}{4}x\right)^2 + \frac{4 \times 3 \times 2}{3!}\left(\frac{3}{4}x\right)^3 + \frac{4 \times 3 \times 2 \times 1}{4!}\left(\frac{3}{4}x\right)^4\right]$$

$$= 4\left(1 + 3x + \frac{27}{8}x^2 + \frac{27}{16}x^3 + \frac{81}{256}x^4\right)$$

$$= 4 + 12x + \frac{27}{2}x^2 + \frac{27}{4}x^3 + \frac{81}{64}x^4$$

Example

Expand $\left(x - \frac{4}{x}\right)^3$.

This can be rewritten as $\sum_{r=0}^{3}\binom{3}{r}x^{3-r}(-1)^r\left(\frac{4}{x}\right)^r$.

Before expanding, it is often useful to simplify this further.

So $\sum_{r=0}^{3}\binom{3}{r}x^{3-r}(-1)^r\left(\frac{4}{x}\right)^r = \sum_{r=0}^{3}\binom{3}{r}x^{3-r}4^r(-1)^rx^{-r}$

$$= \sum_{r=0}^{3}\binom{3}{r}4^r(-1)^rx^{3-2r}$$

Expanding gives

$$\left(x - \frac{4}{x}\right)^3 = x^3 + \binom{3}{1}4^1(-1)^1x^1 + \binom{3}{2}4^2(-1)^2x^{-1} + 4^3(-1)^3x^{-3}$$

$$= x^3 - 12x + 48x^{-1} - 64x^{-3}$$

Example

What is the coefficient of x^2 in the expansion of $\left(3x - \frac{2}{5x}\right)^8$?

Rewriting using sigma notation,

$$\left(3x - \frac{2}{5x}\right)^8 = \sum_{r=0}^{8}\binom{8}{r}3^{8-r}x^{8-r}(-1)^r\left(\frac{2}{5}\right)^r(x^{-1})^r$$

$$= \sum_{r=0}^{8}\binom{8}{r}3^{8-r}(-1)^r\left(\frac{2}{5}\right)^r x^{8-2r}$$

For the x^2 term, it is clear that $8 - 2r = 2$

$$\Rightarrow r = 3$$

Hence the term required is $\binom{8}{3}3^5(-1)^3\left(\frac{2}{5}\right)^3x^2$.

So the coefficient is $56 \times 243 \times (-1) \times \frac{8}{125}$

$$= -\frac{108\,864}{125}$$

This method of finding the required term is very useful, and avoids expanding large expressions.

Example

Find the term independent of x in the expansion of $(2 + x)\left(2x + \frac{1}{x}\right)^5$.

For this to produce a term independent of x, the expansion of $\left(2x + \frac{1}{x}\right)^5$ must have a constant term or a term in x^{-1}.

$$\left(2x + \frac{1}{x}\right)^5 = \sum_{r=0}^{5} 2^{5-r}x^{5-r}(x^{-1})^r$$

So the power of x is given by $5 - 2r$. This cannot be zero for positive integer values of r. Hence the required coefficient is given by

$5 - 2r = -1$

$\Rightarrow r = 3$

The required term is therefore given by $(2 + x)(\cdots + 10 \times 2^2 \times x^{-1} + \ldots)$.

So the term independent of x is 40.

Example

Find the term independent of x in the expansion of $(2x + 1)^7\left(x - \frac{2}{x}\right)^5$.

This is the product of two expansions, which need to be considered separately at first.

$$(2x + 1)^7 = \sum_{r=0}^{7}\binom{7}{r}2^{7-r}x^{7-r}1^r \text{ and } \left(x - \frac{2}{x}\right)^5 = \sum_{k=0}^{5}\binom{5}{k}x^{5-k}(-1)^k2^kx^{-k}$$

So the general terms are $\binom{7}{r}2^{7-r}x^r$ and $\binom{5}{k}(-1)^k2^kx^{5-2k}$.

For the term independent in x, that is x^0, the general terms need to multiply together to make x^0.

So $x^r \cdot x^{5-2k} = x^0$

$\Rightarrow r + 5 - 2k = 0$

$\Rightarrow 2k = r + 5$

This type of equation is often best solved using a tabular method (there is often more than one solution).

k	$2k$	$r + 5$	r
0	0	5	0
1	2	6	1
2	4	7	2
3	6	8	3
4	8	9	4
5	10	10	5
		11	6
		12	7

So the three scenarios that give terms independent of x when the brackets are multiplied are:

$k = 3, r = 1$

$\binom{7}{1}2^6 \times \binom{5}{3}(-1)^3 2^3$
$= 7 \times 64 \times 10 \times -1 \times 8$
$= -35\,840$

$k = 4, r = 2$

$\binom{7}{2}2^5 \times \binom{5}{2}(-1)^2 2^2$
$= 21 \times 32 \times 10 \times 1 \times 4$
$= 26\,880$

$k = 5, r = 5$

$\binom{7}{5}2^2 \times \binom{5}{5}(-1)^5 2^5$
$= 21 \times 4 \times 1 \times -1 \times 32$
$= -2688$

So the term independent of x in the expansion is $-35\,840 + 26\,880 - 2688$
$= -11\,648$

Example

Expand $(2 + x)^5$ and hence find 1.9^5.

$(2 + x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

So 1.9^5 can be considered to be when $x = -0.1$ in the above expansion.

So $1.9^5 = 32 + 80(-0.1) + 80(-0.1)^2 + 40(-0.1)^3 + 10(0.1)^4 + (-0.1)^5$
$= 32 - 8 + 0.8 - 0.04 + 0.001 - 0.00001$
$= 24.76099$

Exercise 8

1 Use the binomial theorem to expand the following expressions.

a $(a + b)^4$ **b** $(3x + 2)^6$ **c** $(1 - x)^4$ **d** $(2p - 3q)^5$

2 Expand the following using the binomial theorem.

a $\left(x + \frac{1}{x}\right)^3$ **b** $\left(x + \frac{2}{x}\right)^5$ **c** $\left(x - \frac{1}{x}\right)^6$ **d** $\left(2t - \frac{1}{4t}\right)^4$

3 Expand $(1 + 3x + x^2)^3$ by considering it as $([1 + 3x] + x^2)^3$.

4 What is the coefficient of:

a x^3 in the expansion of $(x + 2)^5$

b x^5 in the expansion of $(x + 5)^8$

c x^2 in the expansion of $(x - 4)^6$

d x^3 in the expansion of $(2x + 9)^5$

e x in the expansion of $(8 - x)^9$

f x^3 in the expansion of $\left(x + \frac{1}{x}\right)^7$

g x^2 in the expansion of $\left(x - \frac{2}{x}\right)^4$

h the term independent of x in the expansion of $\left(2x - \frac{3}{x}\right)^8$.

5 What is the coefficient of:

a x^3 in the expansion of $(x + 1)^5(2x + 1)^4$

b x^6 in the expansion of $(x - 2)^4(x + 4)^6$

c x in the expansion of $(3 + x)^3(1 - 2x)^5$

d x^2 in the expansion of $(x^2 + x - 3)^4$.

6 Expand these expressions.

a $(x + 5)^3(x - 4)^4$ **b** $\left(x + \frac{1}{x}\right)^3(x - 2)^3$ **c** $\left(x + \frac{1}{x}\right)^4\left(x - \frac{2}{x}\right)^3$

7 What is the coefficient of:

a x in the expansion of $(x + 1)^4\left(x + \frac{1}{x}\right)^3$

b x^3 in the expansion of $(2x + 3)^5\left(x - \frac{1}{x}\right)^4$

c x^{-10} in the expansion of $\left(x + \frac{1}{x}\right)^7\left(x - \frac{4}{x}\right)^5$.

8 Find the term independent of p in the expansion of $\left(2p^2 - \frac{1}{p}\right)^5\left(p + \frac{2}{p}\right)^4$.

9 Calculate the following correct to three significant figures.

a 1.01^4 **b** 0.8^7 **c** 7.94^8

10 For small values of x, any terms with powers higher than 3 are negligible for the expression $(x^2 - x + 5)^2(x - 2)^7$.

Find the approximate expression, $ax^2 + bx + c$, for this expansion.

11 In the expansions $\left(px + \frac{q}{x}\right)^6$ and $\left(px^2 - \frac{q}{x}\right)^4$, the constant terms are equal.

Show that this is never true for $p, q \in \mathbb{R}, p, q \neq 0$.

Review exercise

1 Find the coefficient of x^3 in the binomial expansion of $\left(1 - \frac{1}{2}x\right)^8$.

[IB Nov 02 P1 Q3]

2 The nth term u_n of a geometric sequence is given by $u_n = 3(4)^{n+1}, n \in \mathbb{Z}^+$.

a Find the common ratio r.

b Hence, or otherwise, find S_n, the sum of the first n terms of this sequence.

[IB May 01 P1 Q7]

3 Consider the arithmetic series $2 + 5 + 8 + \cdots$.

a Find an expression for S_n, the sum of the first n terms.

b Find the value of n for which $S_n = 1365$. [IB May 02 P1 Q1]

4 A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the value of

a the common ratio

b the first term. [IB May 03 P1 Q1]

5 The sum of the first n terms of a series is given by $S_n = 2n^2 - n$, where $n \in \mathbb{Z}^+$.

a Find the first three terms of the series.

b Find an expression for the nth term of the series, giving your answer in terms of n. [IB Nov 04 P1 Q3]

6 Consider the infinite geometric series

$$1 + \left(\frac{2x}{3}\right) + \left(\frac{2x}{3}\right)^2 + \left(\frac{2x}{3}\right)^3 + \ldots$$

a For what values of x does the series converge?

b Find the sum of the series if $x = 1.2$. [IB Nov 01 P1 Q4]

7 An arithmetic sequence has 5 and 13 as its first two terms respectively.

a Write down, in terms of n, an expression for the n^{th} term, a_n.

b Find the number of terms of the sequence which are less than 400.

[IB Nov 99 P1 Q1]

8 The coefficient of x in the expansion of $\left(x + \frac{1}{ax^2}\right)^7$ is $\frac{7}{3}$. Find the possible values of a.

[IB Nov 00 P1 Q12]

9 The sum of an infinite geometric sequence is $13\frac{1}{2}$, and the sum of the first three terms is 13. Find the first term. [IB Nov 00 P1 Q15]

10 a $x + 1, 3x + 1, 6x - 2$ are the first three terms of an arithmetic sequence. For what value of n does S_n, the sum of the first n terms, first exceed 100?

b The sum of the first three terms of a positive geometric sequence is 315 and the sum of the 5th, 6th and 7th terms is 80 640. Identify the first term and the common ratio.

11 The first four terms of an arithmetic sequence are $2, a - b, 2a + b + 7$ and $a - 3b$, where a and b are constants. Find a and b. [IB Nov 03 P1 Q9]

12 a Find the expansion of $(2 + x)^5$, giving your answer in ascending powers of x.

b By letting $x = 0.01$ or otherwise, find the **exact** value of 2.01^5.

[IB Nov 04 P1 Q8]

13 The first three terms of a geometric sequence are also the first, eleventh and sixteenth terms of an arithmetic sequence.

The terms of the geometric sequence are all different.

The sum to infinity of the geometric sequence is 18.

a Find the common ratio of the geometric sequence, clearly showing all working.

b Find the common difference of the arithmetic sequence.

[IB May 05 P2 Q4]

14 a An arithmetic progression is such that the sum of the first 8 terms is 424, and the sum of the first 10 terms is 650. Find the fifth term.

b A 28.5 m length of rope is cut into pieces whose lengths are in arithmetic progression with a common difference of d m. Given that the lengths of the shortest and longest pieces are 1 m and 3.75 m respectively, find the number of pieces and the value of d.

c The second and fourth terms of a geometric progression are 24 and 3.84 respectively. Given that all terms are positive, find

i the sum, to the nearest whole number, of the first 5 terms

ii the sum to infinity.

15 Determine the coefficients of $\frac{1}{x}$ and $\frac{1}{x^3}$ in the expansion $\left(2x + \frac{1}{x}\right)^7$.

16 The constant in the expansions of $\left(kx^2 + \frac{6}{x^2}\right)^4$ and $\left(kx^3 + \frac{p}{x^3}\right)^6$ are equal, and k and p are both greater than zero. Express k in terms of p.

17 Find the constant term in the expansion of $\left(3x^2 + \frac{2}{x^6}\right)^{12}$ giving your answer as an integer.

18 What is the coefficient of x^9 in the expansion of $(x^2 - 2x + 1)^3\left(3x + \frac{2}{x}\right)^5$?

19 Simplify $\sum_{k=1}^{n} 6 - 5k^2$ and hence find $\sum_{k=1}^{8} 6 - 5k^2$.

20 Solve $\binom{n+1}{n-2} = 165$.