4 Polynomials

The Italian mathematician Paolo Ruffini, born in 1765, is responsible for synthetic division, also known as Ruffini's rule, a technique used for the division of polynomials that is covered in this chapter.

Paolo Ruffini

Ruffini was not merely a mathematician but also held a licence to practise medicine. During the turbulent years of the French Revolution, Ruffini lost his chair of mathematics at the university of Modena by refusing to swear an oath to the republic. Ruffini seemed unbothered by this, indeed the fact that he could no longer teach mathematics meant that he could devote more time to his patients, who meant a lot to him. It also gave him a chance to do further mathematical research. The project he was working on was to prove that the quintic equation cannot be solved by radicals. Before Ruffini, no other mathematician published the fact that it was not possible to solve the quintic equation by radicals. For example, Lagrange in his paper *Reflections on the resolution of algebraic equations* said that he would return to this question, indicating that he still hoped to solve it by radicals. Unfortunately, although his work was correct, very few mathematicians appeared to care about this new finding. His article was never accepted by the mathematical community, and the theorem is now credited to being solved by Abel.

4.1 Polynomial functions

Polynomials are expressions of the type $f(x) = ax^n + bx^{n-1} + \ldots + px + c$. These expressions are known as polynomials only when all of the powers of x are positive integers (so no roots, or negative powers). The **degree** of a polynomial is the highest power of x (or whatever the variable is called). We are already familiar with some of these functions, and those with a small degree have special names:

This chapter treats this topic as if a calculator is not available throughout until the section on using a calculator at the end.

Degree	Form of polynomial	Name of function
1	$ax + b$	Linear
2	$ax^2 + bx + c$	Quadratic
3	$ax^3 + bx^2 + cx + d$	Cubic
4	$ax^4 + bx^3 + cx^2 + dx + e$	Quartic
5	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f$	Quintic

$f(x) = 2x^5 + 3x^2 - 7$ is a polynomial is of degree 5 or quintic function. The coefficient of the leading term is 2, and -7 is the constant term.

Values of a polynomial

We can evaluate a polynomial in two different ways. The first method is to substitute the value into the polynomial, term by term, as in the example below.

This was covered in Chapter 3.

Example

Find the value of $f(x) = x^3 - 3x^2 + 6x - 4$ when $x = 2$.

Substituting: $f(2) = 2^3 - 3(2)^2 + 6(2) - 4$

$= 8 - 12 + 12 - 4$

$= 4$

The second method is to use what is known as a **nested scheme**.

This is where the coefficients of the polynomial are entered into a table, and then the polynomial can be evaluated, as shown in the example below.

Example

Using the nested calculation scheme, evaluate the polynomial $f(x) = 2x^4 - 4x^3 + 5x - 8$ when $x = -2$.

This needs to be here as there is no x^2 term.

-2	2	-4	0	5	-8
	↓	+	+	+	+
		-4	16	-32	54
	2	-8	16	-27	$\boxed{46}$

Each of these is then multiplied by -2 to give the number diagonally above.

So $f(-2) = 46$

To see why this nested calculation scheme works, consider the polynomial $2x^3 + x^2 - x + 5$.

x	2	1	-1	5
	↓	+	+	+
		$2x$	$2x^2 + x$	$2x^3 + x^2 - x$
	2	$2x + 1$	$2x^2 + x - 1$	$\boxed{2x^3 + x^2 - x + 5}$

Example

Find the value of the polynomial $g(x) = x^3 - 7x + 6$ when $x = 2$.

2	1	0	-7	6
	↓	+	+	+
		2	4	-6
	1	2	-3	$\boxed{0}$

Here $g(2) = 0$. This means that $x = 2$ is a root of $g(x) = x^3 - 7x + 6$.

Division of polynomials

This nested calculation scheme can also be used to divide a polynomial by a linear expression. This is known as **synthetic division**.

When we divide numbers, we obtain a quotient and a remainder. For example, in the calculation $603 \div 40 = 15 \text{ R } 3$, 603 is the dividend, 40 is the divisor, 15 is the quotient and 3 is the remainder.

The same is true for algebraic division. Synthetic division is a shortcut for dividing polynomials by linear expressions – algebraic long division is covered later in the chapter.

Synthetic division works in exactly the same way as the nested calculation scheme. The value of x that is used is the root that the divisor provides. This is best demonstrated by example.

Synthetic division works only for linear divisors.

Example

Divide $3x^3 - x^2 + 2x - 5$ by $x - 2$ using synthetic division.

We need the value of x such that $x - 2 = 0$, that is, $x = 2$.

2	3	-1	2	-5
	↓	+	+	+
		6	10	24
	3	5	12	$\boxed{19}$
	x^2	x		

These numbers are the coefficients of the quotient.

This is the remainder.

So $3x^3 - x^2 + 2x - 5 = (x - 2)(3x^2 + 5x + 12) + 19$

This could be checked by expanding the brackets.

Example

Divide $x^3 - 11x + 3$ by $x + 5$.

-5	1	0	-11	3
	↓	+	+	+
		-5	25	-70
	1	-5	14	$\boxed{-67}$

So $x^3 - 11x + 3 = (x + 5)(x^2 - 5x + 14) - 67$

Example

Divide $2x^3 + x^2 + 5x - 1$ by $2x - 1$.

Here the coefficient of x in the divisor is not 1.

$$2x - 1 = 0$$
$$\Rightarrow 2\left(x - \frac{1}{2}\right) = 0$$
$$\Rightarrow x = \frac{1}{2}$$

$$
\begin{array}{c|cccc}
\frac{1}{2} & 2 & 1 & 5 & -1 \\
 & \downarrow & +1 & +1 & +3 \\
\hline
 & 2 & 2 & 6 & \boxed{2}
\end{array}
$$

So, from this we can say that

$$2x^3 + x^2 + 5x - 1 = \left(x - \frac{1}{2}\right)(2x^2 + 2x + 6) + 2$$
$$= (2x - 1)(x^2 + x + 3) + 2$$

In this situation, there will always be a common factor in the quotient. This common factor is the coefficient of x in the divisor.

Exercise 1

1 Evaluate $f(x) = x^4 - 3x^3 + 3x^2 + 7x - 4$ for $x = 2$.

2 Evaluate $g(x) = 7x^3 - 2x^2 - 8x + 1$ for $x = -2$.

3 Evaluate $f(x) = x^6 - 4x^3 - 7x + 9$ for $x = -1$.

4 Find f(4) for each polynomial.

a $f(x) = x^3 - x^2 + 2x - 5$ **b** $f(x) = 5x^4 - 4x^2 + 8$

c $f(t) = t^5 - 6t^3 + 7t + 6$ **d** $f(x) = 6 - 7x + 5x^2 - 2x^3$

5 Calculate $f\left(-\frac{1}{2}\right)$ for $f(x) = 6x^3 - 4x^2 - 2x + 3$.

6 Use synthetic division to find the quotient and remainder for each of these calculations.

a $(x^2 + 6x - 3) \div (x - 2)$ **b** $(x^3 - 4x^2 + 5x - 1) \div (x - 1)$

c $(2x^3 + x^2 - 8x + 7) \div (x - 6)$ **d** $(x^3 + 5x^2 - x - 9) \div (x + 4)$

e $(x^4 - 5x^2 + 3x + 7) \div (x + 1)$ **f** $(x^5 - x^2 - 5x - 11) \div (2x - 1)$

g $(t^3 - 7t + 9) \div (2t + 1)$ **h** $(-3x^4 - 4x^3 - 5x^2 + 13) \div (4x + 3)$

7 Express each function in the form $f(x) = (px - q)Q(x) + R$ where Q(x) is the quotient on dividing f(x) by $(px - q)$ and R is the remainder.

	f(x)	$(px - q)$
(a)	$3x^2 - 7x + 2$	$x - 2$
(b)	$x^3 + 6x^2 - 8x + 7$	$x - 5$
(c)	$4x^3 + 7x^2 - 9x - 17$	$x + 3$
(d)	$5x^5 - 4x^3 + 3x - 2$	$x + 4$
(e)	$2x^6 - 5x^4 + 9$	$x + 1$
(f)	$x^3 - 7x^2 + 4x - 2$	$2x - 1$
(g)	$2x^4 - 4x^2 + 11$	$2x + 1$

4.2 Factor and remainder theorems

The remainder theorem

If a polynomial f(x) is divided by $(x - h)$ the remainder is f(h).

Proof

We know that $f(x) = (x - h)Q(x) + R$ where Q(x) is the quotient and R is the remainder.

For $x = h$,
$$f(h) = (h - h)Q(h) + R$$
$$= (0 \times Q(h)) + R$$
$$= R$$

Therefore, $f(x) = (x - h)Q(x) + f(h)$.

The factor theorem

If $f(h) = 0$ then $(x - h)$ is a factor of f(x).
Conversely, if $(x - h)$ is a factor of f(x) then $f(h) = 0$.

Proof

For any function $f(x) = (x - h)Q(x) + f(h)$.

If $f(h) = 0$ then $f(x) = (x - h)Q(x)$.

Hence $(x - h)$ is a factor of f(x).

Conversely, if $(x - h)$ is a factor of f(x) then $f(x) = (x - h)Q(x)$.

Hence $f(h) = (h - h)Q(h) = 0$.

Example

Show that $(x + 5)$ is a factor of $f(x) = 2x^3 + 7x^2 - 9x + 30$.

This can be done by substituting $x = -5$ into the polynomial.

$$f(-5) = 2(-5)^3 + 7(-5)^2 - 9(-5) + 30$$
$$= -250 + 175 + 45 + 30$$
$$= 0$$

Since $f(-5) = 0$, $(x + 5)$ is a factor of $f(x) = 2x^3 + 7x^2 - 9x + 30$.

This can also be done using synthetic division. This is how we would proceed if asked to fully factorise a polynomial.

Example

Factorise fully $g(x) = 2x^4 + x^3 - 38x^2 - 79x - 30$.

Without a calculator, we need to guess a possible factor of this polynomial. Since the constant term is -30, we know that possible roots are $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$.

We may need to try some of these before finding a root. Normally we would begin by trying the smaller numbers.

1	2	1	−38	−79	−30
	↓	+	+	+	+
		2	3	−35	−112
	2	3	−35	−112	−142

Clearly $x - 1$ is not a factor.

Trying $x = -1$ and $x = 2$ also does not produce a value of 0. So we need to try another possible factor. Try $x + 2$.

−2	2	1	−38	−79	−30
	↓	+	+	+	+
		−4	6	64	30
	2	−3	−32	−15	0

So $x + 2$ is a factor.

Now we need to factorise $2x^3 - 3x^2 - 32x - 15$. We know that $x = \pm1$ do not produce factors so we try $x = -3$.

−3	2	−3	−32	−15
	↓	+	+	+
		−6	27	15
	2	−9	−5	0

Hence $g(x) = (x + 3)(x + 2)(2x^2 - 9x - 5)$
$= (x + 3)(x + 2)(2x + 1)(x - 5)$

We do not need to use division methods to factorise a quadratic.

Exercise 2

1 Show that $x - 3$ is a factor of $x^2 + x - 12$.
2 Show that $x - 3$ is a factor of $x^3 + 2x^2 - 14x - 3$.
3 Show that $x - 2$ is a factor of $x^3 - 3x^2 - 10x + 24$.
4 Show that $2x - 1$ is a factor of $2x^3 + 13x^2 + 17x - 12$.
5 Show that $3x + 2$ is a factor of $3x^3 - x^2 - 20x - 12$.
6 Show that $x + 5$ is a factor of $x^4 + 8x^3 + 17x^2 + 16x + 30$.

7 Which of these are factors of $x^3 - 28x - 48$?
a $x + 1$ **b** $x - 2$
c $x + 2$ **d** $x - 6$
e $x - 8$ **f** $x + 4$

8 Factorise fully:
a $x^3 - x^2 - x + 1$ **b** $x^3 - 7x + 6$
c $x^3 - 4x^2 - 7x + 10$ **d** $x^4 - 1$
e $2x^3 - 3x^2 - 23x + 12$ **f** $2x^3 + 21x^2 + 58x + 24$
g $12x^3 + 8x^2 - 23x - 12$ **h** $x^4 - 7x^2 - 18$
i $2x^5 + 6x^4 + 7x^3 + 21x^2 + 5x + 15$
j $36x^5 + 132x^4 + 241x^3 + 508x^2 + 388x - 80$

4.3 Finding a polynomial's coefficients

Sometimes the factor and remainder theorems can be utilized to find a coefficient of a polynomial. This is demonstrated in the following examples.

Example

Find p if $x + 3$ is a factor of $x^3 - x^2 + px + 15$.

Since $x + 3$ is a factor, we know that -3 is a root of the polynomial.

Hence the value of the polynomial is zero when $x = -3$ and so we can use synthetic division to find the coefficient.

−3	1	−1	p	15
	↓	+	+	+
		−3	12	−15
	1	−4	$p + 12$	0

This is working backwards from the zero.

So $-3(p + 12) = -15$
$\Rightarrow -3p - 36 = -15$
$\Rightarrow -3p = 21$
$\Rightarrow p = -7$

This can also be done by substitution.
If $f(x) = x^3 - x^2 + px + 15$, then $f(3) = (-3)^3 - (-3)^2 - 3p + 15 = 0$.

So $-27 - 9 - 3p + 15 = 0$
$\Rightarrow -3p - 21 = 0$
$\Rightarrow p = -7$

Example

Find p and q if $x + 5$ and $x - 1$ are factors of $f(x) = 2x^4 + 3x^3 + px^2 + qx + 15$, and hence fully factorise the polynomial.

Using synthetic division for each factor, we can produce equations in p and q.

−5	2	3	p	q	15
	↓	+	+	+	+
		−10	35	$-5p - 175$	−15
	2	−7	$p + 35$	3	0

So $q - 5p - 175 = 3$
$\Rightarrow q = 5p + 178$

1	2	3	p	q	15
	↓	+	+	+	+
		2	5	$p + 5$	−15
	2	5	$p + 5$	−15	0

So $q + p + 5 = -15$
$\Rightarrow q + p = -20$

Solving $q = 5p + 178$ and $q + p = -20$ simultaneously:

$5p + 178 + p = -20$
$\Rightarrow 6p = -198$
$\Rightarrow p = -33$

and $q - 33 = -20$
$\Rightarrow q = 13$

So $f(x) = 2x^4 + 3x^3 - 33x^2 + 13x + 15$

Now we know that $x + 5$ and $x - 1$ are factors:

−5	2	3	−33	13	15
	↓	+	+	+	+
		−10	35	−10	−15
	2	−7	2	3	0

1	2	−7	2	3
	↓	+	+	+
		2	−5	−3
	2	−5	−3	0

Hence $f(x) = (x + 5)(x - 1)(2x^2 - 5x - 3)$
$\Rightarrow f(x) = (2x + 1)(x + 5)(x - 1)(x - 3)$

Exercise 3

1. Find the remainder when $x^3 + 2x^2 - 6x + 5$ is divided by $x + 2$.
2. Find the remainder when $5x^4 + 6x^2 - x + 7$ is divided by $2x - 1$.
3. Find the value of p if $(x - 2)$ is a factor of $x^3 - 3x^2 - 10x + p$.
4. Find the value of k if $(x + 5)$ is a factor of $3x^4 + 15x^3 - kx^2 - 9x + 5$.
5. Find the value of k if $(x - 3)$ is a factor of $f(x) = 2x^3 - 9x^2 + kx - 3$ and hence factorize $f(x)$ fully.
6. Find the value of a if $(x + 2)$ is a factor of $g(x) = x^3 + ax^2 - 9x - 18$ and hence factorize $g(x)$ fully.
7. When $x^4 - x^3 + x^2 + px + q$ is divided by $x - 1$, the remainder is zero, and when it is divided by $x + 2$, the remainder is 27. Find p and q.
8. Find the value of k if $(2x + 1)$ is a factor of $f(x) = 2x^3 + 5x^2 + kx - 24$ and hence factorize $f(x)$ fully.
9. Find the values of p and q if $(x + 3)$ and $(x + 7)$ are factors of $x^4 + px^3 + 30x^2 + 11x + q$.
10. The same remainder is found when $2x^3 + kx^2 + 6x + 31$ and $x^4 - 3x^2 - 7x + 5$ are divided by $x + 2$. Find k.

4.4 Solving polynomial equations

In Chapter 2 we solved quadratic equations, which are polynomial equations of degree 2. Just as with quadratic equations, the method of solving other polynomial equations is to make the polynomial equal to 0 and then factorize.

Example

Solve $x^3 + 4x^2 + x - 6 = 0$.

In order to factorise the polynomial, we need a root of the equation. Here the possible roots are $\pm1, \pm2, \pm3, \pm6$. Trying $x = 1$ works:

1	1	4	1	−6
	↓	+	+	+
		1	5	6
	1	5	6	0

As the remainder is zero, $x - 1$ is a factor.

Hence the equation becomes $(x - 1)(x^2 + 5x + 6) = 0$
$\Rightarrow (x - 1)(x + 3)(x + 2) = 0$
$\Rightarrow x - 1 = 0$ or $x + 3 = 0$ or $x + 2 = 0$
$\Rightarrow x = 1$ or $x = -3$ or $x = -2$

Example

Find the points of intersection of the curve $y = 2x^3 - 3x^2 - 9x + 1$ and the line $y = 2x - 5$.

At intersection, $2x^3 - 3x^2 - 9x + 1 = 2x - 5$

$\Rightarrow 2x^3 - 3x^2 - 11x + 6 = 0$

Here the possible roots are $\pm1, \pm2, \pm3, \pm6$. Trying $x = 3$ works:

3	2	−3	−11	6
	↓	+	+	+
		6	9	−6
	2	3	−2	0

So the equation becomes $(x - 3)(2x^2 + 3x - 2) = 0$

$\Rightarrow (x - 3)(2x - 1)(x + 2) = 0$

$\Rightarrow x = 3 \text{ or } x = \frac{1}{2} \text{ or } x = -2$

To find the points of intersection, we need to find the y-coordinates.

When $x = 3$ | When $x = \frac{1}{2}$ | When $x = -2$

$y = 2(3) - 5 = 1$ | $y = 2\left(\frac{1}{2}\right) - 5 = -4$ | $y = 2(-2) - 5 = -9$

Hence the points of intersection are $(3,1), \left(\frac{1}{2}, -4\right), (-2, -9)$.

Example

Solve $x^3 - 2x^2 - 5x + 6 \le 0$ for $x \ge 0$.

This is an inequality which we can solve in the same way as an equation.

First we need to factorise $x^3 - 2x^2 - 5x + 6$.

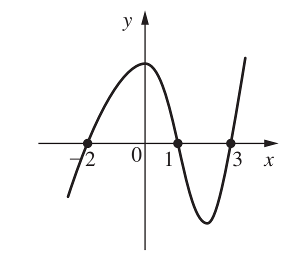
1	1	−2	−5	6
	↓	+	+	+
		1	−1	−6
	1	−1	−6	0

$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$

$= (x - 1)(x + 2)(x - 3)$

So the inequality becomes $(x - 1)(x + 2)(x - 3) \le 0$.

Plotting these points on a graph, and considering points either side of them such as $x = -3, x = 0, x = 2, x = 4$, we can sketch the graph.



This provides the solution: $1 \le x \le 3$

Exercise 4

1 Show that 5 is a root of $x^3 - x^2 - 17x - 15 = 0$ and hence find the other roots.

2 Show that 2 is a root of $2x^3 - 15x^2 + 16x + 12 = 0$ and hence find the other roots.

3 Show that -3 is a root of $2x^5 - 9x^4 - 34x^3 + 111x^2 + 194x - 120 = 0$ and hence find the other roots.

4 Solve the following equations.

a $x^3 - 6x^2 + 5x + 12 = 0$ **b** $x^3 + 7x^2 - 4x - 28 = 0$

c $x^3 + 17x^2 + 75x + 99 = 0$ **d** $x^4 - 4x^3 - 19x^2 + 46x + 120 = 0$

e $x^4 - 4x^3 - 12x^2 + 32x + 64 = 0$ **f** $2x^3 - 13x^2 - 26x + 16 = 0$

g $12x^3 - 16x^2 - 7x + 6 = 0$

5 Find where the graph of $f(x) = x^3 - 8x^2 - 11x + 18$ cuts the x-axis.

6 Show that $x^3 - x^2 + 2x - 2 = 0$ has only one root.

7 Find the only root of $x^3 - 3x^2 + 5x - 15 = 0$.

8 $x = 2$ is a root of $g(x) = x^3 - 10x^2 + 31x - p = 0$.

a Find the value of p.

b Hence solve the equation $g(x) = 0$.

9 $x = -6$ is a root of $f(x) = 2x^3 - 3x^2 - kx + 42 = 0$.

a Find the value of k.

b Hence solve the equation $f(x) = 0$.

10 Solve the following inequalities for $x \ge 0$.

a $(x + 5)(x - 1)(x - 7) \le 0$

b $x^3 - 4x^2 - 11x + 30 \le 0$

c $x^3 + 7x^2 + 4x - 12 \le 0$

d $x^3 - 9x^2 + 11x + 31 \le 10$

e $-6x^3 + 207x^2 + 108x - 105 \ge 0$

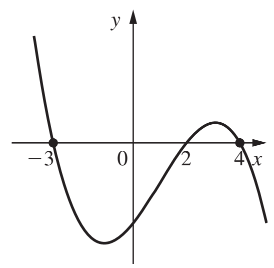
11 The profit of a football club after a takeover is modelled by $P = t^3 - 14t^2 + 20t + 120$, where t is the number of years after the takeover. In which years was the club making a loss?

4.5 Finding a function from its graph

We can find an expression for a function from its graph using the relationship between its roots and factors.

Example

For the graph below, find an expression for the polynomial f(x).



We can see that the graph has roots at $x = -3$, $x = 2$ and $x = 4$.

Hence f(x) has factors $(x + 3)$, $(x - 2)$ and $(x - 4)$.

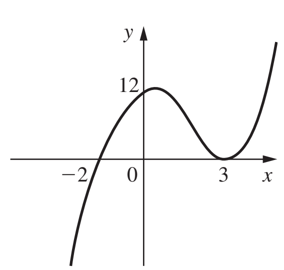
Since the graph cuts the y-axis at $(0, -12)$, we can find an equation:

$$f(x) = k(x + 3)(x - 2)(x - 4)$$
$$f(0) = k(3)(-2)(-4) = -12$$
$$\Rightarrow 24k = -12$$
$$\Rightarrow k = -\frac{1}{2}$$

Hence $f(x) = -\frac{1}{2}(x + 3)(x - 2)(x - 4)$

Example

For the graph below, find an expression for the polynomial f(x).



We can see that the graph has roots at $x = -2$ and $x = 3$.

Hence f(x) has factors $(x + 2)$ and $(x - 3)$. Since the graph has a turning point at (3,0), this is a repeated root (in the same way as quadratic functions that have a turning point on the x-axis have a repeated root).

Since the graph cuts the y-axis at (0,12), we can find the equation:

$$f(x) = k(x + 2)(x - 3)^2$$
$$f(0) = k(2)(-3)^2 = 12$$
$$\Rightarrow 18k = 12$$
$$\Rightarrow k = \frac{2}{3}$$

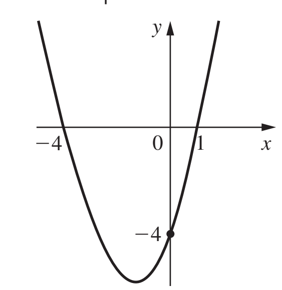
Hence $f(x) = \frac{2}{3}(x + 2)(x - 3)^2$

$$= \frac{2}{3}x^3 - \frac{8}{3}x^2 - 2x + 12$$

Exercise 5

Find an expression for the polynomial f(x) for each of these graphs.

1



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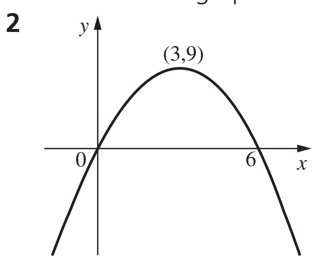

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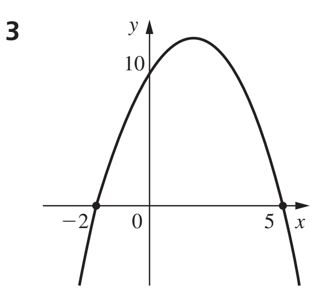

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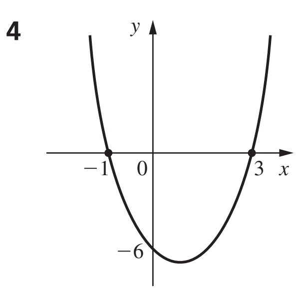

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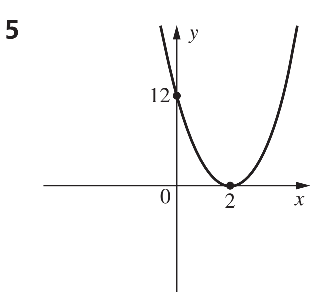

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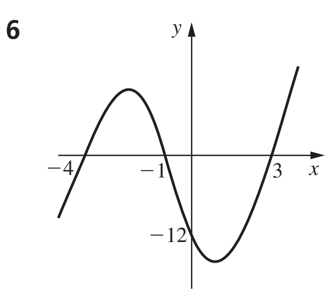

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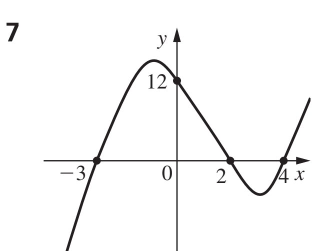

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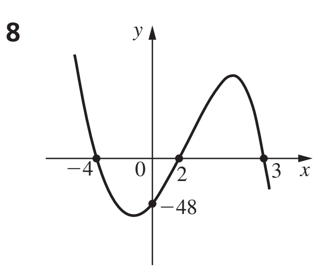

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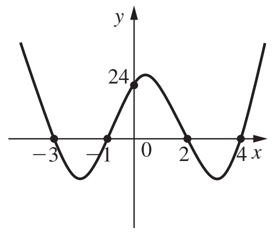

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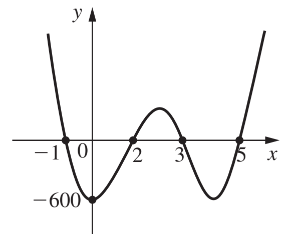

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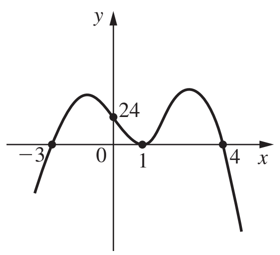


4.6 Algebraic long division

Synthetic division works very well as a "shortcut" for dividing polynomials when the divisor is a linear function. However, it can only be used for this type of division, and in order to divide a polynomial by another polynomial (not of degree 1) it is necessary to use algebraic long division.

This process is very similar to long division for integers.

Example

Find $6087 \div 13$.

$$\begin{array}{r} 4 \\ 13\overline{)6087} \\ \underline{52} \\ 8 \end{array}$$

There are four 13s in 60 remainder 8

This process continues:

$$\begin{array}{r} 468 \\ 13\overline{)6087} \\ \underline{52} \\ 88 \\ \underline{78} \\ 107 \\ \underline{104} \\ 3 \end{array}$$

There are four 13s in 60 remainder 8

There are six 13s in 88 remainder 10

There are eight 13s in 107 remainder 3

Hence $6087 \div 13 = 468 \text{ R } 3$.

This can also be expressed as $6087 = 13 \times 468 + 3$.

To perform algebraic division, the same process is employed.

Example

Find $\dfrac{2x^3 - 7x^2 + 6x - 4}{x - 3}$.

We put this into a division format:

$(x-3)\overline{)2x^3 - 7x^2 + 6x - 4}$

The first step is to work out what the leading term of the divisor x needs to be multiplied by to achieve $2x^3$. The answer to this is $2x^2$ and so this is the first part of the quotient.

$$\begin{array}{rr} & 2x^2\phantom{{}-7x^2+6x-4} \\ (x-3)\!\!\!\! & \overline{)2x^3 - 7x^2 + 6x - 4} \\ & \underline{2x^3 + 6x^2}\phantom{{}+6x-4} \\ & -13x^2 + 6x - 4 \end{array}$$

Multiplying the divisor by $2x^2$ provides this. This is then subtracted from the dividend.

This process is then repeated: the next part of the quotient is what x needs to be multiplied by to give $-13x^2$.

This is continued thus:

$$\begin{array}{rr} & 2x^2 - 13x - 33 \\ (x-3)\!\!\!\! & \overline{)2x^3 - 7x^2 + 6x - 4} \\ & \underline{2x^3 + 6x^2}\phantom{{}+6x-4} \\ & -13x^2 + 6x - 4 \\ & \underline{-13x^2 + 39x}\phantom{{}-4} \\ & -33x - 4 \\ & \underline{-33x + 99} \\ & -103 \end{array}$$

So the remainder is -103.

So $\dfrac{2x^3 - 7x^2 + 6x - 4}{x - 3} = 2x^2 - 13x - 33 - \dfrac{103}{x-3}$.

Example

Find $\dfrac{3x^4 - 4x^3 + 5x^2 - 7x + 4}{x^2 + 2}$.

$$\begin{array}{rr} & 3x^2\phantom{{}-4x^3+5x^2-7x+4} \\ (x^2+2)\!\!\!\! & \overline{)3x^4 - 4x^3 + 5x^2 - 7x + 4} \\ & \underline{3x^4 \phantom{{}-4x^3} + 6x^2}\phantom{{}-7x+4} \\ & -4x^3 - x^2 - 7x + 4 \end{array}$$

To obtain $3x^4$, x^2 must be multiplied by $3x^2$.

This continues to give:

$$
\begin{array}{rl}
 & 3x^2 - 4x - 1 \\
(x^2 + 2) \overline{)} & 3x^4 - 4x^3 + 5x^2 - 7x + 4 \\
 & \underline{3x^4 \qquad\quad + 6x^2} \\
 & - 4x^3 - x^2 - 7x + 4 \\
 & \underline{- 4x^3 \qquad\quad - 8x} \\
 & - x^2 + x + 4 \\
 & \underline{- x^2 \qquad - 2} \\
 & x + 6
\end{array}
$$

x^2 is multiplied by $-4x$ to obtain $-4x^3$. x^2 is multiplied by -1 to obtain $-x^2$.

So the remainder here is $x + 6$.

Hence $\dfrac{3x^4 - 4x^3 + 5x^2 - 7x + 4}{x^2 + 2} = (3x^2 - 4x - 1)(x^2 + 2) + x + 6.$

Although the process of algebraic long division is not part of this curriculum, it is an important skill that is employed in curve sketching, found in Chapter 8.

Example

Find $\dfrac{5 + 3x^3 + x^6}{x^2 + 4}$.

In this case, the numerator is not presented with the powers in descending order. It is vital that it is rearranged so that its powers are in descending order before dividing. There are also some "missing" powers. These must be put into the dividend with zero coefficients to avoid mistakes being made.

$(x^2 + 4)\overline{)x^6 + 0x^5 + 0x^4 + 3x^3 + 0x^2 + 0x + 5}$

Having presented the division as above, the process is the same.

$$
\begin{array}{rl}
 & x^4 \qquad - 4x^2 + 3x + 16 \\
(x^2 + 4) \overline{)} & x^6 + 0x^5 + 0x^4 + 3x^3 + 0x^2 + 0x + 5 \\
 & \underline{x^6 \qquad + 4x^4} \\
 & - 4x^4 + 3x^3 + 0x^2 + 0x + 5 \\
 & \underline{- 4x^4 \qquad - 16x^2} \\
 & 3x^3 + 16x^2 + 0x + 5 \\
 & \underline{3x^3 \qquad + 12x} \\
 & 16x^2 - 12x + 5 \\
 & \underline{16x^2 \qquad + 64} \\
 & -12x - 59
\end{array}
$$

Hence $\dfrac{5 + 3x^3 + x^6}{x^2 + 4} = x^4 - 4x^2 + 3x + 16 - \dfrac{12x + 59}{x^2 + 4}.$

Exercise 6

Use algebraic long division for the following questions.

1 Show that $\dfrac{x^2 + 5x + 6}{x + 2} = x + 3.$

2 Show that $(x^2 - 1) \div (x - 1) = x + 1.$

3 Find $(x^2 - x - 12) \div (x - 4).$

4 Find $\dfrac{2x^2 - 5x - 3}{x - 3}.$

5 Find the quotient and remainder for $\dfrac{3x^2 - 5x - 7}{x + 2}.$

6 Show that $x = 5$ is a root of $x^3 - 5x^2 + 4x - 20 = 0.$

7 Find $\dfrac{x^3 + 5x^2 - 9x + 4}{x^2 + 5}$, expressing your answer in the form $px + q + \dfrac{ax + b}{x^2 + 5}.$

8 Find $\dfrac{2x^3 - 9x^2 - 8x + 11}{2x^2 + 7}.$

9 Find $\dfrac{3x^4 - 2x^2 + 7}{x + 2}.$

10 Find $\dfrac{2x^5 + 13}{x^2 - 3}.$

11 Find $\dfrac{9 - 5x + 7x^2 - x^4}{2x + 1}.$

12 Find $\dfrac{11 + 5x^3 - x^6}{x^2 + 1}.$

13 Given that $x - 1$ is a factor of $f(x)$, solve $f(x) = x^4 + 4x^3 - 7x^2 - 22x + 24 = 0.$

14 Given that $x^2 + x + 4$ is a factor of $g(x)$, solve $g(x) = x^5 + 7x^4 + 6x^3 - 4x^2 - 40x - 96 = 0.$

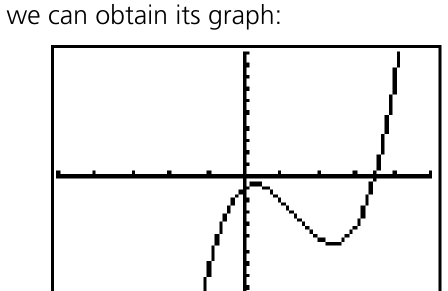
4.7 Using a calculator with polynomials

Everything covered so far in this chapter has been treated as if a calculator were not available. For the polynomials examined so far it has been possible to factorise and hence solve polynomial equations to find the roots. For many polynomials it is not possible to factorise, and the best method of solving these is to employ graphing calculator technology. In these cases, the roots are not exact values and so are given as approximate roots. Also, if a calculator is available, it can help to factorise a polynomial as it removes the need for trial and error to find the initial root.

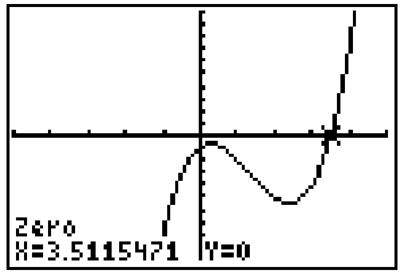
Example

Sketch the graph of $f(x) = x^3 - 4x^2 + 2x - 1$ and hence find its root(s).

Using a calculator, we can obtain its graph:



We can see that this cubic function has only one root. We can find this using the calculator. As the calculator uses a numerical process to find the root, it is important to make the left bound and right bound as close as possible to the root.

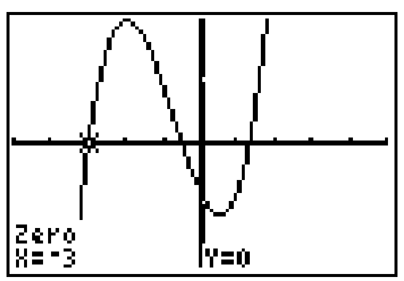


Hence $x = 3.51$.

Example

Factorise fully $f(x) = 6x^3 + 13x^2 - 19x - 12$.

Using the graph of the function,



we can see that $x = -3$ is a root of $f(x)$ and so we can use this in synthetic division.

−3	6	13	−19	−12
	↓	+	+	+
		−18	15	12
	6	−5	−4	$\boxed{0}$

Hence $f(x) = (x + 3)(6x^2 - 5x - 4)$
$= (x + 3)(3x - 4)(2x + 1)$.

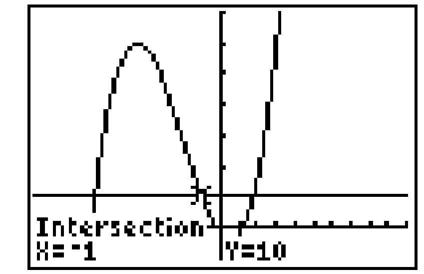
Solving a polynomial inequality is also made simple through the use of graphing calculator technology.

Example

(a) Solve $f(x) < 10$ where $f(x) = x^3 + 6x^2 - 7x - 2$.

(b) Find the range of values of a so that there are three solutions to the equation $f(x) = a$.

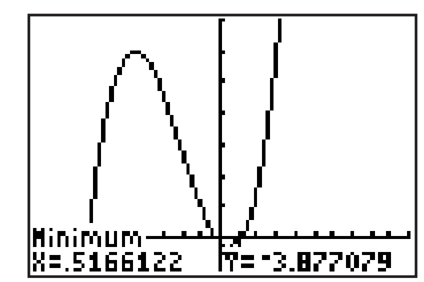
(a) Here is the graph of $f(x)$:



We can find the three points of intersection of $f(x)$ with the line $y = 10$: $x = -6.77, -1, 1.77$

Looking at the graph, it is clear that the solution to the inequality is $x < -6.77$ and $-1 < x < 1.77$.

(b) By calculating the maximum and minimum turning points, we can find the values of a so that there are three solutions. To have three solutions, a must lie between the maximum and minimum values (y-values).



From the calculator, it is clear that $-3.88 < a < 59.9$.

Exercise 7

Use a calculator to solve all of the following equations.

1 $2x^3 - 5x^2 - 4x + 3 = 0$

2 $2x^4 + 9x^3 - 46x^2 - 81x - 28 = 0$

3 $2x^4 + 3x^3 - 15x^2 - 32x - 12 = 0$

4 $3x^3 - 9x^2 + 4x - 12 = 0$

5 $x^3 + 2x^2 - 11x - 5 = 0$

6 $x^3 - 6x^2 + 4x - 7 = 0$

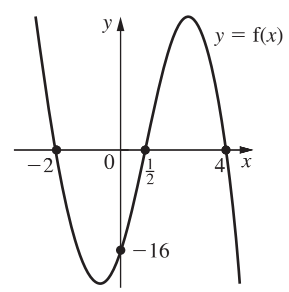
7 $x^4 - x^3 + 5x^2 - 7x + 2 = 0$

8 $x^4 - 22x^2 - 19x + 41 = 0$

9 $x^3 - 9x^2 - 11x + 4 = 2x + 1$

10 For what value of x do the curves $y = 2x^3 - 4x^2 + 3x + 7$ and $y = x^2 - 3x - 5$ meet?

11 State the equation of f(x) from its graph below. Hence find the points of intersection of f(x) and $g(x) = x^2 - 4$.



12 Factorise $x^3 - 21x + 20$, by obtaining an initial root using a calculator.

13 Factorise $36x^4 - 73x^2 + 16$, by obtaining an initial root using a calculator.

14 **a** Solve $f(x) = x^3 + 6x^2 - 7x - 2 < 0$.

b Find the values of a so that there are three solutions to the equation $f(x) = a$.

15 **a** Solve $f(x) = x^3 + 6x^2 - 7x - 2 < 20$.

b Find the values of a so that there are three solutions to the equation $f(x) = a$.

Review Exercise

1 Evaluate $2x^3 - 5x^2 + 3x + 7$ when $x = -2$.

2 Find $(3x^4 - 2x^3 + 6x + 1) \div (x - 2)$.

3 Express $f(x) = 2x^5 + 4x^2 - 7$ in the form $Q(x)(2x - 1) + R$ by dividing f(x) by $2x - 1$.

4 Show that $x + 2$ is a factor of $x^3 - 10x^2 + 3x + 54$, and hence find the other factors.

5 Factorise fully $f(x) = 2x^3 - 3x^2 - 17x - 12$.

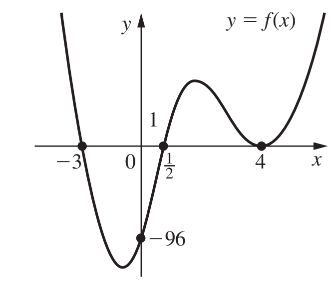
6 Factorise fully $g(x) = 6x^4 - 19x^3 - 59x^2 + 16x + 20$.

7 Factorise fully $k(x) = x^4 - 3x^3 + x^2 - 15x - 20$.

8 Show that $x = -2$ is a root of $x^3 + 6x^2 - 13x - 42 = 0$, and hence find the other roots.

9 Solve $2x^5 + 5x^4 + x^3 + 34x^2 - 66x + 24 = 0$.

10 Find an expression for f(x) from its graph.



11 Solve $5x^3 - 6x + 7 = 0$, correct to 3 significant figures.

12 Solve $x^3 - 7x^2 - 2x + 31 = 0$, correct to 3 significant figures.

13 Using algebraic long division, find $\dfrac{x^3 - 5x^2 + 6x - 4}{2x + 1}$.

14 Using algebraic long division, find $(x^4 - 4x^2 + 3x - 5) \div (x^2 + 1)$.

15 The polynomial $x^3 + ax^2 - 3x + b$ is divisible by $(x - 2)$ and has a remainder 6 when divided by $(x + 1)$. Find the value of a and of b. [IB May 03 P1 Q4]

16 The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $x - 2$ as when divided by $x + 1$. Find the value of a. [IB Nov 01 P1 Q3]

17 When the polynomial $x^4 + ax + 3$ is divided by $x - 1$, the remainder is 8. Find the value of a. [IB Nov 02 P1 Q1]

18 Consider $f(x) = x^3 - 2x^2 - 5x + k$. Find the value of k if $x + 2$ is a factor of f(x). [IB Nov 04 P1 Q1]