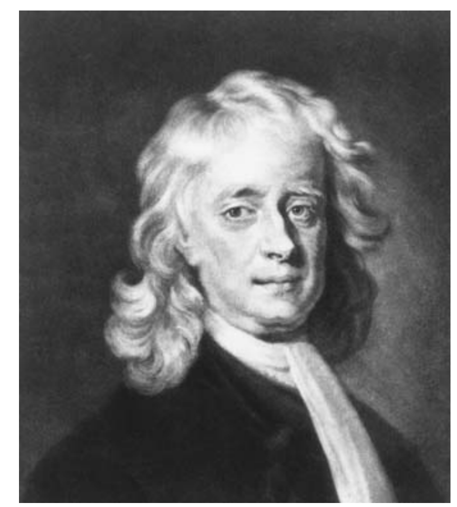
8 Differential Calculus 1 – Introduction

The ideas that are the basis for calculus have been with us for a very long time. Between 450 BC and 225 BC, Greek mathematicians were working on problems that would find their absolute solution with the invention of calculus. However, the main developments were much more recent; it was not until the 16th century that major progress was made by mathematicians such as Fermat, Roberval and Cavalieri. In the 17th century, calculus as it is now known was developed by Sir Isaac Newton and Gottfried Wilhelm von Leibniz.

Sir Isaac Newton famously "discovered" gravity when an apple fell on his head.



Sir Isaac Newton

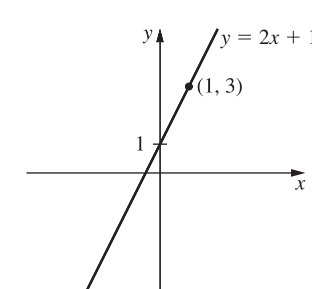
Gottfried Wilhelm von Leibniz

Consider the graph of a quadratic, cubic or trigonometric function.

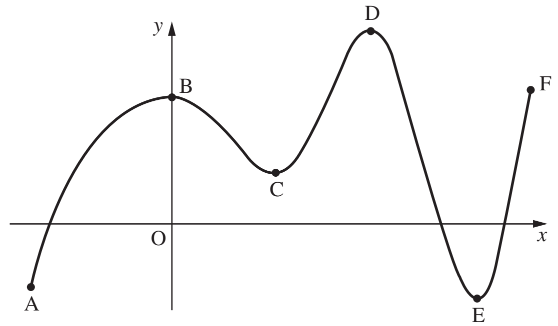
Differential calculus is a branch of mathematics that is concerned with rate of change. In a graph, the rate of change is the gradient. Although linear functions have a constant gradient, most functions have changing gradients. Being able to find a pattern for the gradient of curves is the aim of differentiation. Differentiation is the process used to find rate of change.

The gradient of a straight line is constant.

For example, in the diagram below, the $\text{gradient} = 2$.



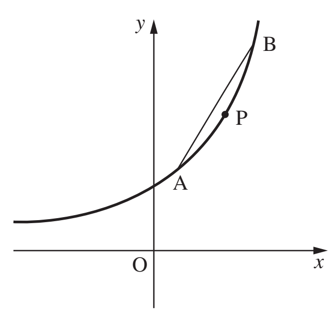
However, when a curve is considered, it is obvious that the gradient is constantly changing.



The sections AB, CD, EF have positive gradient (the function is increasing) and sections BC, DE have negative gradient (the function is decreasing). The question we need to answer is: how do we measure the gradient of a curve?

8.1 Differentiation by first principles

We know gradient $= \dfrac{\Delta y}{\Delta x}$, and one method of finding the gradient of a straight line is to use $\dfrac{y_2 - y_1}{x_2 - x_1}$.



Consider the coordinates $(x, \text{f}(x))$ and $(x + h, \text{f}(x + h))$ – the gap between the x-coordinates is h. This can be used to find an approximation for the gradient at P as seen in the diagram.

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{f}(x + h) - \text{f}(x)}{x + h - x} = \frac{\text{f}(x + h) - \text{f}(x)}{h}$$

This is calculating the gradient of the chord AB shown in the diagram. As the chord becomes smaller, the end-points of the curve are getting closer together, and h becomes smaller. Obviously, this approximation becomes more accurate as h becomes smaller. Finally h becomes close to zero and the chord's length becomes so small that it can be considered to be the same as the point P.

The gradient of a function, known as the **derivative**, with notation $\text{f}'(x)$ is defined:

$$\text{f}'(x) = \lim_{h \to 0} \frac{\text{f}(x + h) - \text{f}(x)}{h}$$

The notation $\lim\limits_{h \to 0}$ means the limit as h tends to zero. This is the value to which the expression converges as h becomes infinitesimally small.

The idea of a limit is similar to sum of a infinite series met in Chapter 6 and also to horizontal asymptotes in Chapter 3.

Example

Find the derivative of $\text{f}(x) = 3x^2$.

$\text{f}(x + h) = 3(x + h)^2 = 3x^2 + 6hx + 3h^2$

$\text{f}(x + h) - \text{f}(x) = 3x^2 + 6hx + 3h^2 - 3x^2 = 6hx + 3h^2$

$$\frac{\text{f}(x + h) - \text{f}(x)}{h} = \frac{6hx + 3h^2}{h} = 6x + 3h$$

So $\text{f}'(x) = \lim\limits_{h \to 0} \dfrac{\text{f}(x + h) - \text{f}(x)}{h} = 6x$ (as $3h \to 0$)

Hence at any point on the curve, the gradient is given by $6x$.

This process is known as differentiation by first principles.

Example

Find the derivative of $\text{f}(x) = x^3$.

$\text{f}(x + h) = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$\text{f}(x + h) - \text{f}(x) = x^3 + 3x^2h + 3xh^2 + h^3 - x^3 = 3x^2h + 3xh^2 + h^3$

$$\frac{\text{f}(x + h) - \text{f}(x)}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2$$

So $\text{f}'(x) = \lim\limits_{h \to 0} \dfrac{\text{f}(x + h) - \text{f}(x)}{h} = 3x^2$ (as $3xh + h^2 \to 0$)

Example

Find the derivative of $\text{f}(x) = -7x$.

$\text{f}(x + h) = -7(x + h) = -7x - 7h$

$\text{f}(x + h) - \text{f}(x) = -7x - 7h - (-7x) = -7h$

$$\frac{\text{f}(x + h) - \text{f}(x)}{h} = \frac{-7h}{h} = -7$$

So $\text{f}'(x) = \lim\limits_{h \to 0} \dfrac{\text{f}(x + h) - \text{f}(x)}{h} = -7$

Example

Find the derivative of $f(x) = 5$.

$f(x+h) = 5$

$f(x+h) - f(x) = 5 - 5 = 0$

$$\frac{f(x+h)-f(x)}{h} = \frac{0}{h} = 0$$

So $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = 0$

What happens in a sum or difference of a set of functions? Consider the sum of the previous three examples, i.e. $f(x) = x^3 - 7x + 5$.

Example

Find the derivative of $f(x) = x^3 - 7x + 5$.

$$\begin{aligned} f(x+h) &= (x+h)^3 - 7(x+h) + 5 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - 7x - 7h + 5 \\ f(x+h) - f(x) &= x^3 + 3x^2h + 3xh^2 + h^3 - 7x - 7h + 5 - x^3 + 7x - 5 \\ &= 3x^2h + 3xh^2 + h^3 - 7h \\ \frac{f(x+h)-f(x)}{h} &= \frac{3x^2h + 3xh^2 + h^3 - 7h}{h} \\ &= 3x^2 + 3xh + h^2 - 7 \end{aligned}$$

So $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = 3x^2 - 7$ (as $3xh + h^2 \to 0$)

This demonstrates that differentiation of a function containing a number of terms can be differentiated term by term.

Exercise 1

Find $f'(x)$ using the method of differentiation from first principles:

1 $f(x) = 5x$ **2** $f(x) = 8x$ **3** $f(x) = -2x$

4 $f(x) = x^2$ **5** $f(x) = x^3$ **6** $f(x) = x^4$

7 $f(x) = 2x^2$ **8** $f(x) = 5x^2$ **9** $f(x) = 4x^3$

10 $f(x) = 9$ **11** $f(x) = \frac{3}{x}$ **12** $f(x) = x^2 + 4$

13 $f(x) = 8 - 3x$ **14** $f(x) = x^2 - 4x + 9$ **15** $f(x) = 2x - \frac{1}{x}$

8.2 Differentiation using a rule

Looking at the patterns in Exercise 1, it should be obvious that for:

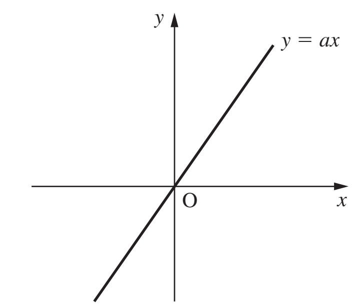
$$f(x) = ax^n$$
$$f'(x) = anx^{n-1}$$

Multiply by the power and subtract 1 from the power

This rule can be used to perform differentiation.

Unless specifically required, differentiation by first principles is not used – the above rule makes the process much shorter and easier.

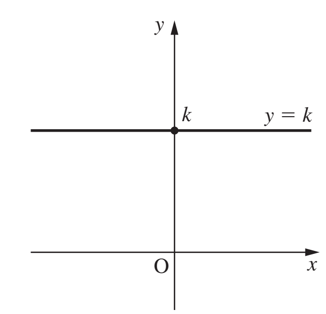
In particular, notice that $f(x) = ax$ gives $f'(x) = a$.



This is no surprise – the gradient of a linear function is constant.

Also, $f(x) = k$ gives $f'(x) = 0$.

The gradient of a horizontal line is zero.



Differential calculus was developed by two mathematicians, Isaac Newton and Gottfried Leibniz. There are two commonly used notations:

Functional / Newtonian notation	Geometrical / Leibniz notation
$f(x) =$	$y =$
Derivative $f'(x) =$	$\frac{dy}{dx} =$

Either notation can be used, and both will appear in questions.

Example

Differentiate $y = 5x^3 - \frac{4}{x}$.

Simplifying, $y = 5x^3 - 4x^{-1}$

$$\frac{dy}{dx} = 5 \cdot 3x^{(3-1)} - (-4)x^{(-1-1)}$$
$$= 15x^2 + 4x^{-2}$$
$$= 15x^2 + \frac{4}{x^2}$$

As with first principles, we can differentiate a sum by differentiating term by term.

Example

Find the derivative of $f(x) = \frac{2}{\sqrt[3]{x}}$.

$$f(x) = 2x^{-\frac{1}{3}}$$
$$f'(x) = -\frac{2}{3}x^{-\frac{4}{3}}$$

Example

Differentiate $y = \frac{2x(x-5)}{\sqrt{x}}$.

$$y = \frac{2x^2 - 10x}{x^{\frac{1}{2}}}$$
$$= 2x^{\frac{3}{2}} - 10x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 2 \cdot \frac{3}{2}x^{\frac{1}{2}} - 10 \cdot \frac{1}{2}x^{-\frac{1}{2}}$$
$$= 3x^{\frac{1}{2}} - 5x^{-\frac{1}{2}}$$

Sometimes it is necessary to simplify the function before differentiating.

Example

Find $g'(4)$ for $g(x) = x^2 - x - 6$.

Here we are evaluating the derivative when $x = 4$.

First differentiate: $g'(x) = 2x - 1$

Then substitute $x = 4$: $g'(4) = 2 \cdot 4 - 1 = 7$

So the gradient of $g(x)$ at $x = 4$ is 7.

Example

Find the coordinates of the points where the gradient is -2 for $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 8x + 7$.

Here we are finding the points on the curve where the derivative is -2.

First differentiate: $f'(x) = x^2 - x - 8$

Then solve the equation $f'(x) = x^2 - x - 8 = -2$

$$\Rightarrow x^2 - x - 6 = 0$$
$$\Rightarrow (x+2)(x-3) = 0$$
$$\Rightarrow x = -2 \text{ or } \Rightarrow x = 3$$

At $x = -2$, $y = \frac{55}{3}$ and at $x = 3$, $y = -\frac{25}{2}$

So the coordinates required are $\left(-2, \frac{55}{3}\right)$ and $\left(3, -\frac{25}{2}\right)$.

Exercise 2

1 Differentiate these functions.

a $f(x) = 9x^2$ **b** $f(x) = 10x^3$ **c** $f(x) = 6x^4$

d $f(x) = -3x^5$ **e** $f(x) = 12$ **f** $f(x) = 7x$

g $f(x) = 11x$ **h** $f(x) = 8x - 9$ **i** $f(x) = \frac{4}{x^2}$

j $f(x) = 5\sqrt{x}$ **k** $y = x^2 + 5x + 6$ **l** $f(x) = \frac{5}{\sqrt{x^5}}$

m $y = x^3 + 5x^2 - 7x - 4$ **n** $y = 6x^2 - \frac{2}{x}$ **o** $y = \sqrt[4]{x}$

p $y = \sqrt[3]{x^5}$ **q** $y = \frac{4x(x^2-3)}{3x^2}$ **r** $y = \frac{3x^2(x^3-3)}{5\sqrt{x}}$

2 Find $f'(3)$ for $f(x) = x^2 - 4x + 9$.

3 Find $g'(6)$ for $g(x) = \frac{4 - x^2}{x}$.

4 Find the gradient of $y = x^3 - 6x + 9$ when $x = 2$.

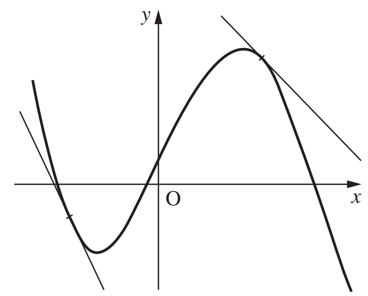
5 Find the gradient of $y = \frac{4x^2 - 9}{\sqrt{x}}$ when $x = 16$.

6 Find the coordinates of the point where the gradient is 4 for $f(x) = x^2 - 6x + 12$.

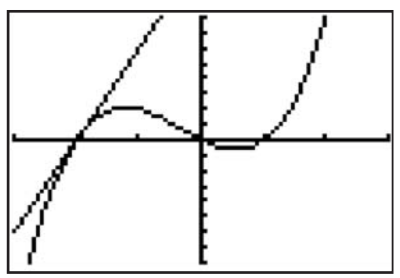
7 Find the coordinates of the points where the gradient is 2 for $f(x) = \frac{2}{3}x^3 - \frac{9}{2}x^2 - 3x + 8$.

8.3 Gradient of a tangent

A tangent is a straight line that touches a curve (or circle) at one point.



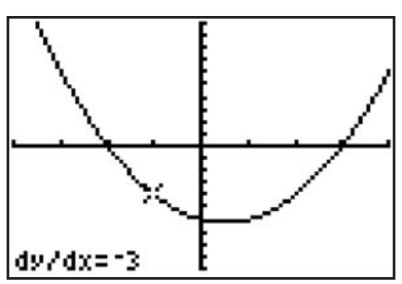
Differentiation can be used to find the value of the gradient at any particular point on the curve. At this instant the value of the gradient of the curve is the same as the gradient of the tangent to the curve at that point.



Finding the gradient using a graphing calculator

Using a graphing calculator, the value of the gradient at any point can be calculated.

For example, for $y = x^2 - x - 6$, at $x = -1$



This is helpful, especially for checking answers. However, we often need the derivative function and so need to differentiate by hand. The calculator can only find the gradient using a numerical process and is unable to differentiate algebraically.

Tangents and normals

The gradient at a point is the same as the gradient of the tangent to the curve at that point. Often it is necessary to find the equation of the tangent to the curve.

Method for finding the equation of a tangent

1. Differentiate the function.
2. Substitute the required value to find the gradient.
3. Find the *y*-coordinate (if not given).
4. Find the equation of the tangent using this gradient and the point of contact using $y - y_1 = m(x - x_1)$.

8 Differential Calculus 1 – Introduction

Example

Find the equation of the tangent to $y = x^2 - x - 6$ at $x = -1$.

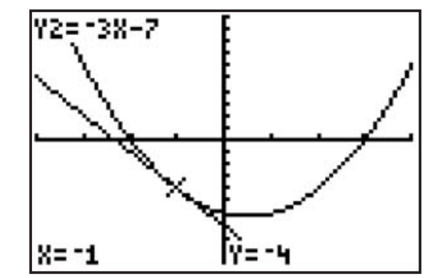
Differentiating,

$\frac{dy}{dx} = 2x - 1$ and so at $x = -1$, $\frac{dy}{dx} = 2 \times (-1) - 1 = -3$

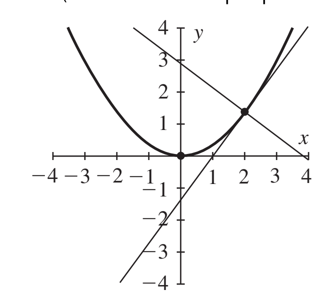
The point of contact is when $x = -1$, and so $y = (-1)^2 - (-1) - 6 = -4$, i.e. $(-1, -4)$

Using $y - y_1 = m(x - x_1)$, the equation of the tangent is

$y - (-4) = -3\ (x - -1) \Rightarrow y = -3x - 7$



The normal to a curve is also a straight line. The normal to the curve is perpendicular to the curve at the point of contact (therefore it is perpendicular to the tangent).



Finding the equation of a normal to a curve is a very similar process to finding the tangent.

Method for finding the equation of a normal

1. Differentiate the function.
2. Substitute the required value to find the gradient.
3. Find the gradient of the perpendicular using $m_1 m_2 = -1$.
4. Find the *y*-coordinate (if not given).
5. Find the equation of the tangent using this gradient and the point of contact using $y - y_1 = m(x - x_1)$.

Example

Find the equation of the tangent, and the equation of the normal, to $y = x^2 - 9x - 12$ at $x = 3$.

Using the method, $\frac{dy}{dx} = 2x - 9$.

At $x = 3$, $\frac{dy}{dx} = 2(3) - 9 = -3$.

At $x = 3$, $y = 3^2 - 9(3) - 12 = -30$.

So the equation of the tangent is $y + 30 = -3(x - 3)$

$$\Rightarrow y = -3x - 21$$

The equation of the normal uses the same point but the gradient is different.

Using $m_1 m_2 = -1$, the gradient of the normal is $\frac{1}{3}$.

Using $y - y_1 = m(x - x_1)$ the equation of the normal is $y + 30 = \frac{1}{3}(x - 3)$

$$\Rightarrow y = \frac{1}{3}x - 31$$

To find the perpendicular gradient turn the fraction upside down and change the sign.

Example

Find the equation of the tangent, and the equation of the normal, to $y = x^3 - 1$ where the curve crosses the x-axis.

The curve crosses the x-axis when $y = 0$. So $x^3 = 1 \Rightarrow x = 1$, i.e. $(1, 0)$

Differentiating, $\frac{dy}{dx} = 3x^2$.

At $x = 1$, $\frac{dy}{dx} = 3(1)^2 = 3$.

Using $y - y_1 = m(x - x_1)$, the equation of the tangent is $y = 3(x - 1) = 3x - 3$.

Then the gradient of the normal will be $-\frac{1}{3}$.

Using $y - y_1 = m(x - x_1)$ the equation of the normal is $y = -\frac{1}{3}(x - 1)$

$$\Rightarrow y = -\frac{1}{3}x + \frac{1}{3}$$

Exercise 3

1 Find the equation of the tangent and the equation of the normal to:

a $y = 3x^2$ at $x = 1$ **b** $y = x^2 - 3x$ at $x = 2$

c $y = x^4$ at $x = -1$ **d** $y = \sqrt{x}$ at $x = 9$

e $y = \frac{4}{x^2}$ at $x = 1$ **f** $y = 20 - 3x^2$ at $x = -3$

2 The curve $y = (x^2 + 3)(x - 1)$ meets the x-axis at A and the y-axis at B. Find the equation of the tangents at A and B.

3 Find the equation of the normal to $y = \frac{16}{x^3}$ at $x = 2$.

4 The tangent at $P(1, 0)$ to the curve $y = x^3 + x^2 - 2$ meets the curve again at Q. Find the coordinates of Q.

5 Find the equation of the tangent to $y = 9 - 2x - 2x^2$ at $x = -1$.

6 Find the equations of the tangents to the curve $y = (2x + 1)(x - 1)$ at the points where the curve cuts the x-axis. Find the point of intersection of these tangents.

7 Find the equations of the tangents to the curve $y = 3x^2 - 5x - 9$ at the points of intersection of the curve and the line $y = 6x - 5$.

8 Find the equation of the normal to $y = x^2 - 3x + 2$, which has a gradient of 3.

9 Find the equations for the tangents at the points where the curves $y = x^2 - x - 6$ and $y = -x^2 + x + 6$ meet.

10 For $y = x^2 - 7$, find the equation of the tangent at $x = 1$.

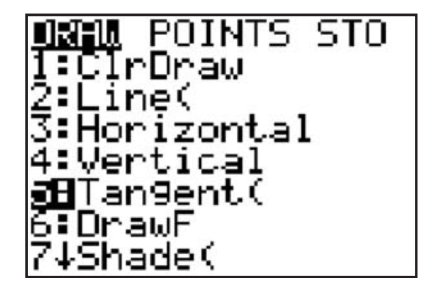
For $y = x^2 - x - 12$, find the equation of the normal to the curve at $x = 4$.

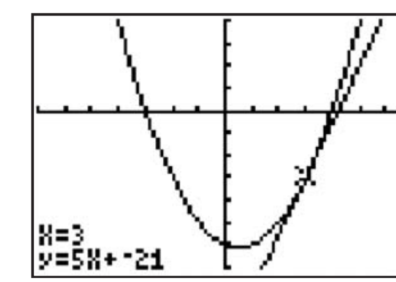
Now find the area of the triangle formed between these two lines and the y-axis.

Tangents on a graphing calculator

It is possible to draw the tangent to a curve using a graphing calculator.

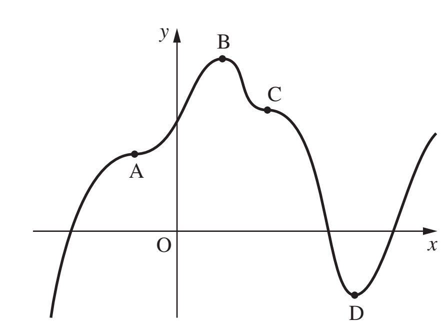
To find the equation of the tangent to $y = x^2 - x - 12$ at $x = 3$, the calculator can draw the tangent and provide the equation of the tangent.





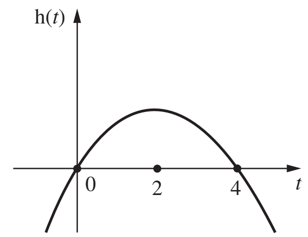
8.4 Stationary points

The gradient of a curve is constantly changing. In some regions, the function is increasing, in others it is decreasing, and at other points it is stationary.



At points A, B, C and D, the tangent to the curve is horizontal and so the gradient is zero. These points are known as **stationary points**. Often these points are very important to find, particularly when functions are used to model real-life situations.

For example, a stone is thrown and its height, in metres, is given by $h(t) = 4t - t^2$, $0 \le t \le 4$.



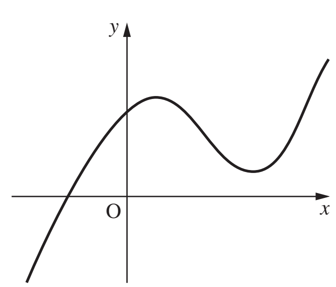
$h'(t) = 4 - 2t$ and so $h'(t) = 0$ when $4 - 2t = 0$

i.e. $t = 2$

So the maximum height of the stone is given by $h(2) = 4$ metres, which is the point where the gradient is zero. We have met this concept before as maximum and minimum turning points in Chapters 2, 3 and 4, and these are in fact examples of stationary points.

- Stationary points are when $\dfrac{dy}{dx} = 0$.
- Stationary points are coordinate points.
- The x-coordinate is when the stationary point occurs.
- The y-coordinate is the stationary value.

Note that the maximum turning point is not necessarily the maximum value of that function. Although it is the maximum value in that region (a local maximum) there may be greater values. For example, for the cubic function $y = x^3 + 2x^2 - 3x + 4$ the greatest value is not a turning point as it tends to infinity in the positive x-direction.



Method for finding stationary points

1. Differentiate the function.
2. Solve the equation $\dfrac{dy}{dx} = 0$.
3. Find the y-coordinate of each stationary point.

Example

Find the stationary points of $y = x^3 - 7x^2 - 5x + 1$.

Differentiating, $\dfrac{dy}{dx} = 3x^2 - 14x - 5$

When $\dfrac{dy}{dx} = 0$, $3x^2 - 14x - 5 = 0$

So $(3x - 1)(x + 5) = 0$

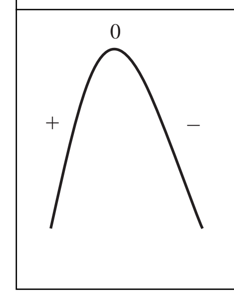
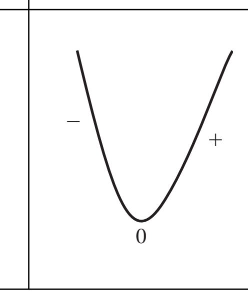
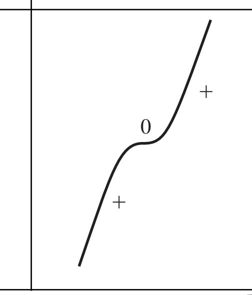
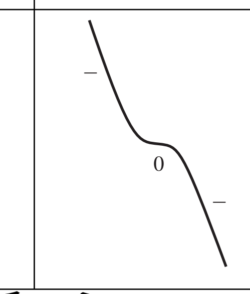
$\Rightarrow x = \dfrac{1}{3}$ or $x = -5$

When $x = \dfrac{1}{3}$, $y = -\dfrac{38}{27}$ and when $x = -5$, $y = -274$

So the stationary points are $\left(\dfrac{1}{3}, -\dfrac{38}{27}\right)$ and $(-5, -274)$.

Determining the nature of stationary points

There are four possible types of stationary point.

Maximum turning point	Minimum turning point	Rising point of inflexion	Falling point of inflexion
			

A stationary point of inflexion is when the sign of the gradient does not change either side of the stationary point.

There are two methods for testing the nature of stationary points.

Method 1 — Using the signs of f′(x)

Here the gradient immediately before and after the stationary point is examined. This is best demonstrated by example.

Example

Find the stationary points of $y = 2x^3 + 3x^2 - 36x + 5$ and determine their nature.

Using the steps of the method suggested above,

1. $\dfrac{dy}{dx} = 6x^2 + 6x - 36$

2. $\dfrac{dy}{dx} = 0$

$\Rightarrow 6x^2 + 6x - 36 = 0$
$\Rightarrow 6(x + 3)(x - 2) = 0$
$\Rightarrow x = -3$ or $x = 2$

3. When $x = -3, y = 2(-3)^3 + 3(-3)^2 - 36(-3) + 5$
$= 86$

When $x = 2, y = 2(2)^3 + 3(2)^2 - 36(2) + 5$
$= -39$

Therefore the coordinates of the stationary points are $(-3, 86)$ and $(2, -39)$.

To find the nature of the stationary points, we can examine the gradient before and after $x = -3$ and $x = 2$ using a table of signs.

This means the negative side of -3

This means the positive side of -3

$x =$	-3^-	-3	-3^+	2^-	2	2^+
$\frac{dy}{dx}$	+	0	–	–	0	+
Shape	/	—	\	\	—	/

We can choose values either side of the stationary point to test the gradient either side of the stationary point. This is the meaning of the notation -3^+ and -3^-. -3^+ means taking a value just on the positive side of –3, that is slightly higher than –3. -3^- means taking a value just on the negative side of –3, that is slightly lower than –3.

It is important to be careful of any vertical asymptotes that create a discontinuity.

So for -3^-, $x = -4$ could be used and so $\frac{dy}{dx} = 6(-4 + 3)(-4 - 2)$. What is important is whether this is positive or negative. The brackets are both negative and so the gradient is positive. A similar process with, say $x = -2, x = 1$, $x = 3$, fills in the above table.

This provides the shape of the curve around each stationary value and hence the nature of each stationary point.

So $(-3, 86)$ is a maximum turning point and $(2, -39)$ is a minimum turning point. Strictly these should be known as a **local maximum** and a **local minimum** as they are not necessarily the maximum or minimum values of the function – these would be called the **global** maximum or minimum.

Example

Find the stationary point for $y = x^3$ and determine its nature.

1. $\frac{dy}{dx} = 3x^2$

2. $\frac{dy}{dx} = 0$ when $3x^2 = 0$

$\Rightarrow x = 0$

8 Differential Calculus 1 – Introduction

3. When $x = 0, y = 0$, i.e. (0, 0)

4.

$x =$	0^-	0	0^+
$\frac{dy}{dx}$	+	0	+
Shape	/	—	/

So the stationary point (0, 0) is a rising point of inflexion.

Method 2 — Using the sign of $f''(x)$

When a function is differentiated a second time, the rate of change of the gradient of the function is found. This is known as the **concavity** of the function.

The two notations used here for the second derivative are:

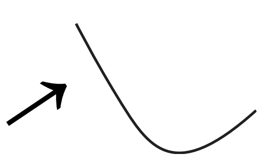
$f''(x)$ in functional notation and $\frac{d^2y}{dx^2}$ in Leibniz notation.

This Leibniz notation arises from differentiating $\frac{dy}{dx}$ again.

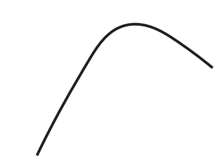
This is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

For a section of curve, if the **gradient** is increasing then it is said to be concave up.

The curve is getting less steep in this section, i.e. it is becoming less negative and so is increasing.

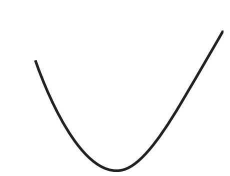


Similarly, if the gradient is decreasing it is said to be concave down.



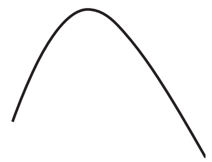
Looking at the sign of $\frac{d^2y}{dx^2}$ can help us determine the nature of stationary points.

Consider a minimum turning point:



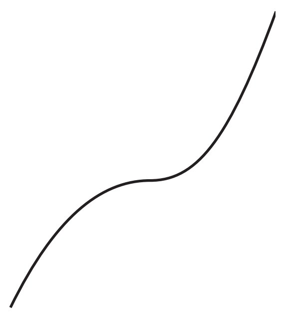
At the turning point, $\frac{dy}{dx} = 0$, although the gradient is zero, the gradient is increasing (moving from negative to positive) and so $\frac{d^2y}{dx^2}$ is positive.

Consider a maximum turning point:

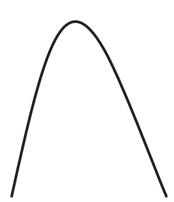
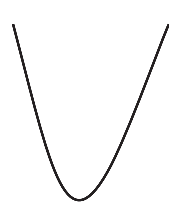
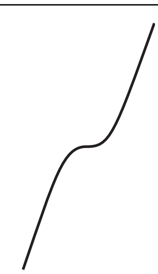
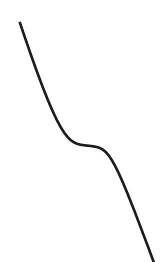


At the turning point, $\frac{dy}{dx} = 0$, although the gradient is zero, the gradient is decreasing (moving from positive to negative) and so $\frac{d^2y}{dx^2}$ is negative.

At a point of inflexion, $\frac{d^2y}{dx^2}$ is zero.



This table summarizes the nature of stationary points in relation $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

	Maximum turning point	**Minimum turning point**	**Rising point of inflexion**	**Falling point of inflexion**
				
$\frac{dy}{dx}$	0	0	0	0
$\frac{d^2y}{dx^2}$	–	+	0	0

This method is often considered more powerful than method 1 (when the functions become more complicated). For examination purposes, it is always best to use the second derivative to test nature. However, note that for stationary points of inflexion, it is still necessary to use a table of signs.

Although the table above is true, it is unfortunately not the whole picture. A positive or negative answer for $\frac{d^2y}{dx^2}$ provides a conclusive answer to the nature of a stationary point.

$\frac{d^2y}{dx^2} = 0$ is not quite as helpful. In most cases, this will mean that there is a stationary point of inflexion. However, this needs to be tested using a table of signs as it is possible that it will in fact be a minimum or maximum turning point. A table of signs is also required to determine whether a stationary point of inflexion is rising or falling. See the second example below for further clarification.

Example

Find the stationary points of $y = x + \frac{4}{x}$.,

1. $\frac{dy}{dx} = 1 - 4x^{-2}$
2. This is stationary when $\frac{dy}{dx} = 0$.

   So $1 - 4x^{-2} = 0$

   $\Rightarrow \frac{4}{x^2} = 1$

   $\Rightarrow x^2 = 4$

   $\Rightarrow x = -2$ or $x = 2$
3. When $x = -2, y = -4$, i.e. $(-2, -4)$ and when $x = 2, y = 4$, i.e. (2, 4)
4. To test the nature using the second derivative,

   $\frac{d^2y}{dx^2} = 8x^{-3} = \frac{8}{x^3}$

   At $x = -2$, $\frac{d^2y}{dx^2} = \frac{8}{-8} = -1$ and since this is negative, this is a maximum turning point.

   At $x = 2$, $\frac{d^2y}{dx^2} = \frac{8}{8} = 1$ and since this is positive, this is a minimum turning point.

   So stationary points are $(-2, -4)$, a local maximum, and (2, 4), a local minimum.

Example

Find the stationary point(s) of $y = x^4$.

1. $\frac{dy}{dx} = 4x^3$
2. Stationary when $\frac{dy}{dx} = 0$,

   So $4x^3 = 0$

   $\Rightarrow x^3 = 0$

   $\Rightarrow x = 0$

3. When $x = 0, y = 0$, i.e. (0, 0).
4. To test the nature using the second derivative,

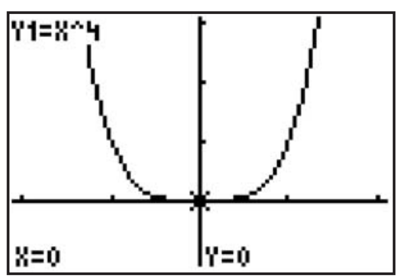
$$\frac{d^2y}{dx^2} = 12x^2$$

At $x = 0, \frac{d^2y}{dx^2} = 12 \times 0^3 = 0$

As $\frac{d^2y}{dx^2} = 0$ for this stationary point, no assumptions can be made about its nature and so a table of signs is needed.

$x =$	0^-	0	0^+
$\frac{dy}{dx}$	$-$	0	$+$
Shape	\	—	/

Hence (0, 0) is a minimum turning point. This can be verified with a calculator.



This is an exceptional case, which does not often occur. However, be aware of this "anomaly".

Exercise 4

1 Find the stationary points and determine their nature using a table of signs.

a $f(x) = x^2 - 8x + 3$

b $y = x^3 - 12x + 7$

c $f(x) = 5x^4$

d $y = (3x - 4)(x + 2)$

e $f(x) = 4x + \frac{1}{x}$

2 Find the stationary points and determine their nature using the second derivative.

a $y = 2x^2 + 8x - 5$

b $y = (4 - x)(x + 6)$

c $f(x) = x(x - 4)^2$

d $y = 2x^3 - 9x^2 + 12x + 5$

e $f(x) = 3x^5$

3 Find the stationary points and determine their nature using either method.

a $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 4$

b $y = (2x - 5)^2$

c $f(x) = 16x - \frac{1}{x^2}$

d $y = x^6$

e $y = x^5 - 2x^3 + 5x^2 + 2$

4 Find the distance between the turning points of the graph of $y = -(x^2 - 4)(x^2 + 2)$.

8.5 Points of inflexion

The concavity of a function is determined by the second derivative.

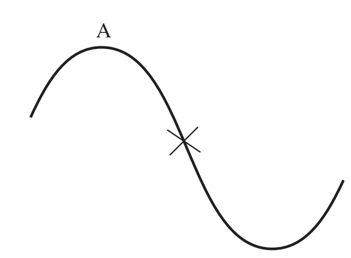
$f''(x) > 0$	Concave up
$f''(x) < 0$	Concave down

So what happens when $f''(x) = 0$?

We know that when $f'(x) = 0$ and $f''(x) = 0$, there is a stationary point – normally a stationary point of inflexion (with the exceptions as previously discussed).

In fact, apart from the previously noted exceptions, whenever $f''(x) = 0$ it is known as a **point of inflexion**. The type met so far are stationary points of inflexion when the gradient is also zero (also known as horizontal points of inflexion).

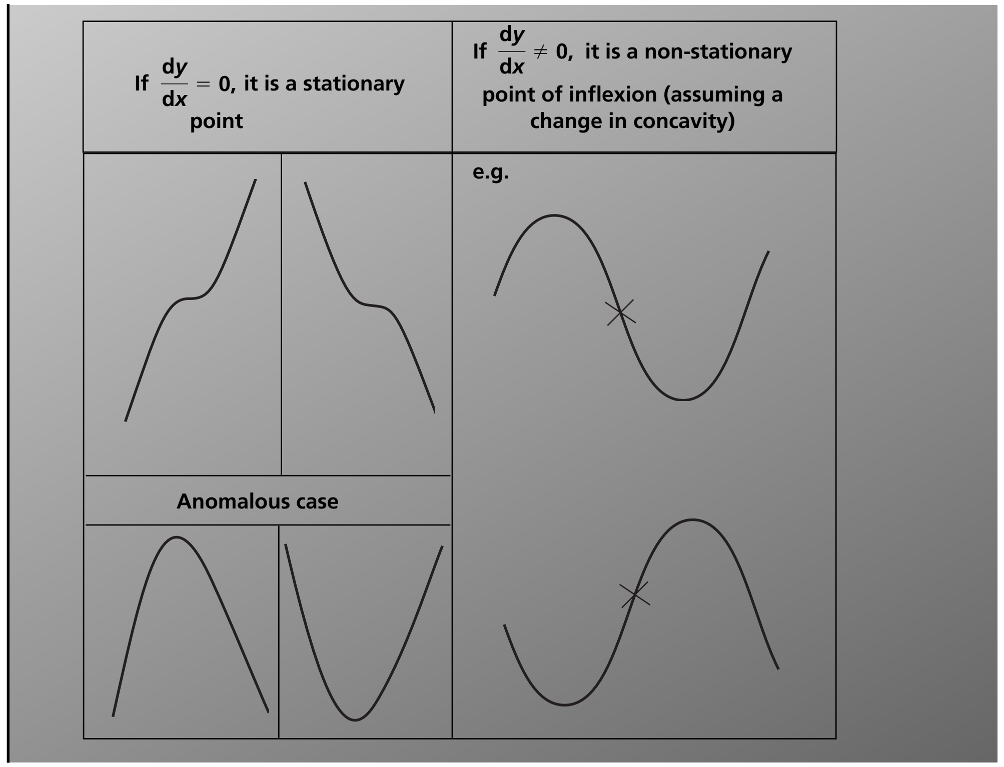
However, consider the curve:



Looking at the gradient between the turning points, it is constantly changing, the curve becoming steeper and then less steep as it approaches B. So the rate of change of gradient is negative (concave down) around A and then positive (concave up) around B.

Clearly the rate of change of gradient $\left(\frac{d^2y}{dx^2}\right)$ must be zero at some point between A and B. This is the steepest part of the curve between A and B, and it is this point that is known as a point of inflexion. This is clearly not stationary. So a point of inflexion can now be defined to be a point where the concavity of the graph changes sign.

If $\frac{d^2y}{dx^2} = 0$, there is a point of inflexion:



Method for finding points of inflexion

1. Differentiate the function twice to find $\frac{d^2y}{dx^2}$.
2. Solve the equation $\frac{d^2y}{dx^2} = 0$.
3. Find the y-coordinate of each point.
4. Test the concavity around this point, i.e. $\frac{d^2y}{dx^2}$ must change sign.

Example

Find the points of inflexion of the curve $f(x) = x^5 - 15x^3$ and determine whether they are stationary.

1. $f'(x) = 5x^4 - 45x^2$

   $f''(x) = 20x^3 - 90x$

2. For points of inflexion, $f''(x) = 0$

   So $20x^3 - 90x = 0$

   $\Rightarrow 10x(2x^2 - 9) = 0$

   $\Rightarrow x = 0$ or $x^2 = \frac{9}{2}$

   $\Rightarrow x = 0$ or $x = \frac{\pm 3}{\sqrt{2}}$

3. $f\left(-\frac{3}{\sqrt{2}}\right) = 100.2324$

   $f(0) = 0$

   $f\left(\frac{3}{\sqrt{2}}\right) = -100.2324$

4.

$x =$	$-\frac{3}{\sqrt{2}}^-$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}^+$	0^-	0	0^+
$f''(x)$	–	0	+	+	0	–

$x =$	$\frac{3}{\sqrt{2}}^-$	$\frac{3}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}^+$
$f''(x)$	–	0	+

There is a change in concavity (the sign of the second derivative changes) around each point. So each of these three points is a point of inflexion.

To test whether each point is stationary, consider $f'(x)$.

$$f'\left(\frac{3}{\sqrt{2}}\right) = 5\left(\frac{3}{\sqrt{2}}\right)^4 - 45\left(\frac{3}{\sqrt{2}}\right)^2 = -\frac{405}{4}$$

Hence $x = \frac{3}{\sqrt{2}}$ provides a non-stationary point of inflexion.

$$f'\left(-\frac{3}{\sqrt{2}}\right) = 5\left(-\frac{3}{\sqrt{2}}\right)^4 - 45\left(-\frac{3}{\sqrt{2}}\right)^2 = -\frac{405}{4}$$

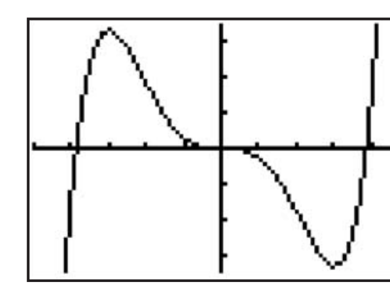
Hence $x = -\frac{3}{\sqrt{2}}$ also provides a non-stationary point of inflexion.

$f'(0) = 5(0)^4 - 45(0)^2 = 0$

Hence $x = 0$ provides a stationary point of inflexion.

The three points of inflexion are $\left(-\frac{3}{\sqrt{2}}, -\frac{405}{4}\right)$, $(0, 0)$ and $\left(\frac{3}{\sqrt{2}}, -\frac{405}{4}\right)$.

This can be verified on a calculator.



Exercise 5

1 Find the points of inflexion for the following functions and determine whether they are stationary.

a $f(x) = x^5 - \frac{40}{3}x^3$

b $f(x) = x^3 + 3x^2 - 6x + 7$

c i $y = x^2 - x + 18$

ii $y = 5x^2 - 9$

iii $f(x) = ax^2 + bx + c$

Make a general statement about quadratic functions.

2 Find the points of inflexion for the following functions and determine whether they are stationary.

a $y = 4x^3$

b $f(x) = x^3 - 3x^2 + 7$

c $y = -x^3 - 6x^2 + 8x - 3$

d $f(x) = ax^3 + bx^2 + cx + d$

Make a general statement about cubic functions.

3 Find the points of inflexion for the following functions and determine whether they are stationary.

a $f(x) = x^4 - 6x^2 + 8$

b $y = 3x^4 + 5x^3 - 3x^2 + 7x + 3$

c $f(x) = x^5 - 3x^4 + 5x^3$

d $y = x^4 - 3$

4 Find the equation of the tangent to $y = x^3 - 9x^2 + 6x + 9$ at the point of inflexion.

5 For the graph of $y = 2x^3 - 12x^2 + 5x - 3$, find the distance between the point of inflexion and the root.

8.6 Curve sketching

Bringing together knowledge of functions, polynomials and differentiation, it is now possible to identify all the important features of a function and hence sketch its curve.

The important features of a graph are:

- **Vertical asymptotes** (where the function is not defined)
  This is usually when the denominator is zero.
- **Intercepts**
  These are when $x = 0$ and $y = 0$.
- **Stationary points and points of inflexion**
  Determine when $\frac{dy}{dx} = 0$ and when $\frac{d^2y}{dx^2} = 0$.
- **Behaviour as** $x \to \pm\infty$
  This provides horizontal and oblique asymptotes.

With the exception of oblique asymptotes, all of the necessary concepts have been met in this chapter, and in Chapters 1, 2, 3 and 4.

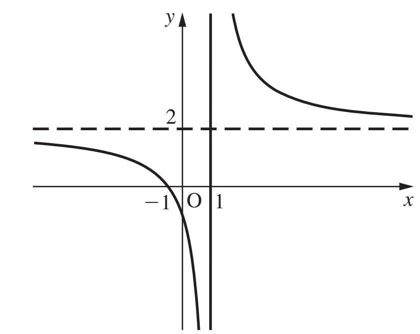
Oblique asymptotes

In Chapter 3 we met horizontal asymptotes. These occur where $x \to \pm\infty$. This is also true for oblique asymptotes.

Consider the function $y = 2 + \frac{3}{x-1}$.

It is clear that as $x \to \pm\infty$, the $\frac{3}{x-1}$ becomes negligible and so $y \to 2$.

Hence $y = 2$ is a horizontal asymptote for this function.



Now consider the function $y = 2x - 1 + \frac{5}{3x+2}$.

In a similar way, the fractional part $\left(\frac{5}{3x+2}\right)$ tends to zero as $x \to \pm\infty$ and so $y \to 2x - 1$. This means that $y = 2x - 1$ is an **oblique asymptote** (also known as a slant asymptote).

Method for sketching a function

1. Find the vertical asymptotes (where the function is not defined).
2. If it is an improper rational function (degree of numerator $\geq$ degree of denominator), divide algebraically to produce a proper rational function.
3. Consider what happens for very large positive and negative values of x. This will provide horizontal and oblique asymptotes.
4. Find the intercepts with the axes.
   These are when $x = 0$ and $y = 0$.
5. Find the stationary points and points of inflexion (and their nature).
   Determine when $\frac{dy}{dx} = 0$ and when $\frac{d^2y}{dx^2} = 0$.
6. Sketch the curve, ensuring that all of the above important points are annotated on the graph.

Example

Find the asymptotes of $y = \frac{-2x^2 - 7x - 1}{x - 4}$.

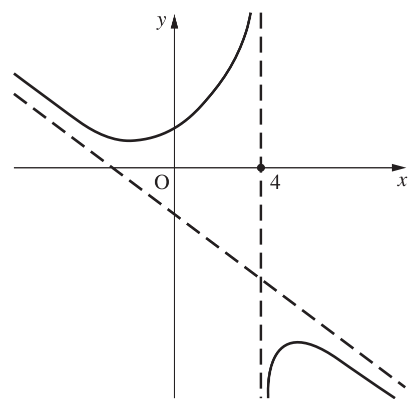
Clearly the function is not defined at $x = 4$ and so this is a vertical asymptote.

Dividing,
$$\begin{array}{r} -2x - 15 \\ (x-4)\overline{)\,-2x^2 - 7x - 1} \\ \underline{-2x^2 + 8x} \\ -15x - 1 \\ \underline{-15x + 60} \\ -61 \end{array}$$

Hence $y = -2x - 15 - \dfrac{61}{x-4}$ and so as $x \to \pm\infty$, $y \to -2x - 15$.

Therefore $y = -2x - 15$ is an oblique asymptote.

This is clear from the graph:



Example

Sketch the graph of $y = \dfrac{x^2}{x+1}$, identifying all asymptotes, intercepts, stationary points, and non-horizontal points of inflexion.

There is a vertical asymptote at $x = -1$

Dividing,

$$\begin{array}{r@{}l} & \;\; x - 1 \\ (x+1)\,\big) & \overline{x^2 \qquad\quad} \\ & \underline{x^2 + x} \\ & \;\; -x \\ & \underline{-x - 1} \\ & \qquad 1 \end{array}$$

This gives $y = x - 1 + \dfrac{1}{x+1}$.

So $y = x - 1$ is an oblique asymptote.

Putting $x = 0$ and $y = 0$ gives an intercept at $(0, 0)$. There are no other roots.

Preparing for differentiation, $y = x - 1 + \dfrac{1}{x+1}$

$$= x - 1 + (x - 1)^{-1}$$

Differentiating, $\dfrac{dy}{dx} = 1 - (x+1)^{-2}$

Stationary when $\dfrac{dy}{dx} = 0$, i.e. when $\dfrac{1}{(x+1)^2} = 1$

$$\Rightarrow x + 1 = -1 \text{ or } x + 1 = 1$$

$$\Rightarrow x = -2 \text{ or } x = 0$$

Substituting into the original function provides the coordinates $(0, 0)$ and $(-2, -4)$.

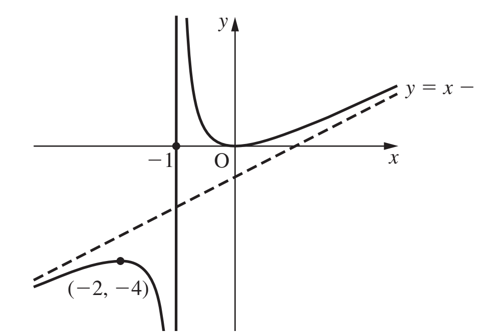
For the nature of these stationary points,

$$\frac{d^2y}{dx^2} = 2(x+1)^{-3} = \frac{2}{(x+1)^3}$$

At $x = 0$, $\dfrac{d^2y}{dx^2} = 2 > 0$ so $(0, 0)$ is a local minimum turning point.

At $x = -2$, $\dfrac{d^2y}{dx^2} = -2 < 0$ so $(-2, -4)$ is a local maximum point.

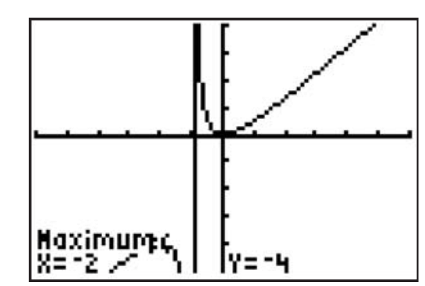
As $\dfrac{d^2y}{dx^2} = \dfrac{2}{(x+1)^3} \neq 0 \;\; \forall x \in \mathbb{R}$, there are no non-horizontal points of inflexion.



The notation $\forall$ means "for all". So $\forall\; x \in \mathbb{R}$, means for all real values of x.

All of these features can be checked using a graphing calculator, if it is available. In some cases, an examination question may expect the use of a graphing calculator to find some of these important points, particularly stationary points.

For example, the calculator provides this graph for the above function:



In fact, it is possible to be asked to sketch a graph that would be difficult without use of the calculator. Consider the next example.

Example

$$\frac{x^2}{9} - \frac{y^2}{4} = 36$$

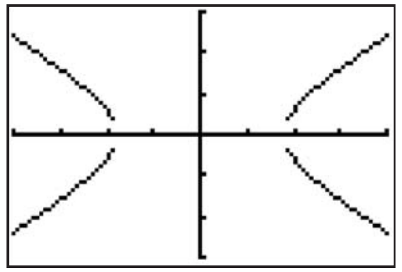
In order to graph this function using a calculator, we need to rearrange into a $y =$ form.

$$\frac{x^2}{9} - \frac{y^2}{4} = 36$$

$$\Rightarrow \frac{y^2}{4} = \frac{x^2}{9} - 36$$

$$\Rightarrow y^2 = \frac{4x^2}{9} - 144$$

$$\Rightarrow y = \pm\sqrt{\frac{4x^2}{9} - 144}$$



Clearly this function is not defined for a large section (in fact $-18 < x < 18$).

The oblique asymptotes are not immediately obvious. Rearranging the equation to give $\frac{y}{x} = \pm\sqrt{\frac{4}{9} - \frac{144}{x^2}}$ makes it clearer.

As $x \to \pm\infty, \frac{y}{x} \to \pm\frac{2}{3}$

$$\Rightarrow y \to \pm\frac{2}{3}x$$

Exercise 6

Find all asymptotes (vertical and non-vertical) for these functions.

1 $y = \frac{2}{x}$ **2** $y = \frac{x}{x-3}$ **3** $y = \frac{x^2+5}{x}$

4 $f(x) = \frac{x+3}{x-2}$ **5** $f(x) = \frac{x^2+1}{x+2}$ **6** $y = \frac{2x^2+3x-5}{x-3}$

7 $f(x) = \frac{x^3-2x^2+3x+5}{x^2+4}$ **8** $y = \frac{x^3-4x}{x^2+1}$

9 $y = \frac{5x}{(x-1)(x-4)}$ **10** $f(x) = \frac{x^2+6}{x^2-1}$

11 $y = \frac{3x^2+8}{x^2-9}$ **12** $y = \frac{4x^3+9}{x^2-x-6}$

Sketch the graphs of these functions, including asymptotes, stationary points and intercepts.

13 $y = \frac{x-1}{x+1}$ **14** $y = \frac{x-1}{x(x+1)}$ **15** $y = \frac{x}{x+4}$

16 $y = \frac{2x^2}{x+1}$ **17** $y = \frac{x}{x^2-1}$ **18** $y = \frac{x^2}{1-x}$

19 $y = \frac{(2x+5)(x-4)}{(x+2)(x-3)}$ **20** $y = \frac{1}{x^2-x-12}$

21 $y = \frac{4}{x^2-2}$ **22** $\frac{x^2}{16} - \frac{y^2}{9} = 25$

8.7 Sketching the graph of the derived function

Given the graph of the original function it is sometimes useful to consider the graph of its derivative. For example, non-horizontal points of inflexion now become obvious from the graph of the derived function, since they become stationary points. Horizontal points of inflexion are already stationary.

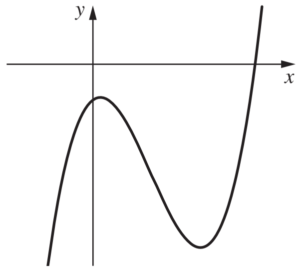
If the original function is known, then it is straightforward to sketch the graph of the derived function. This can be done by:

a) finding the derivative and sketching it

b) using a graphing calculator to sketch the derived function.

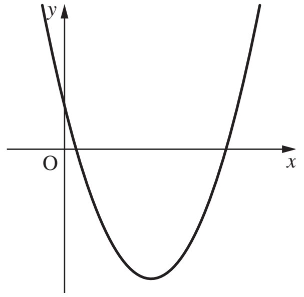
Example

Sketch the derivative of $y = x^3 - 6x^2 + 3x - 5$.

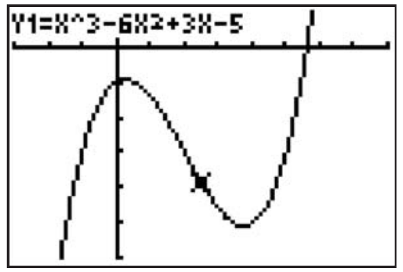


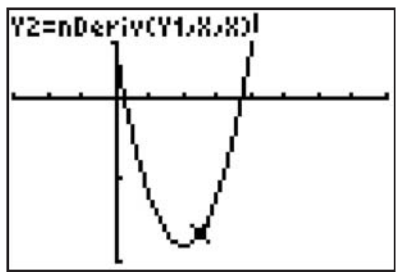
Using method a)

Differentiating gives $\frac{dy}{dx} = 3x^2 - 12x + 3$



Using method b)



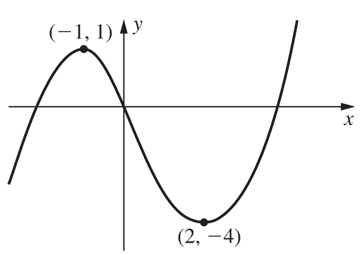


These methods are possible only if the function is known.

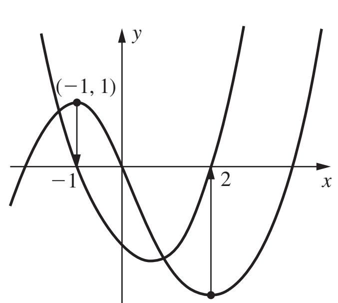
If the function is not known, the gradient of the graph needs to be examined.

Example

Sketch the graph of the derivative of this graph.



Note where the graph is increasing, stationary and decreasing. Stationary points on the original graph become roots of the derived function $\left(\frac{dy}{dx} = 0\right)$, increasing regions are above the x-axis $\left(\frac{dy}{dx} > 0\right)$, and decreasing regions are below the x-axis $\left(\frac{dy}{dx} < 0\right)$. So the graph becomes



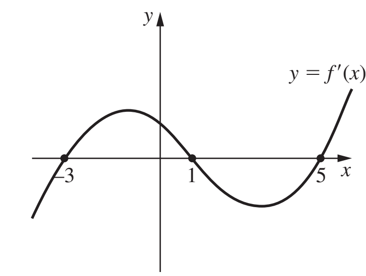
With polynomial functions, the degree of a derived function is always one less than the original function.

We can consider this process in reverse, and draw a possible graph of the original function, given the graph of the derived function. In order to do this note that:

1. roots of the derived function are stationary points on the original graph
2. stationary points on the derived function graph are points of inflexion on the original graph.

Example

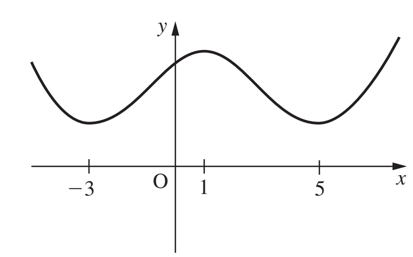
Given this graph of the derived function $y = f'(x)$, sketch a possible graph of the original function $y = f(x)$.



We can see that there are roots on this graph and hence stationary points on the original at $x = -3$, $x = 1$, and $x = 5$.
It is helpful to consider the gradient of the original to be able to draw a curve.

$x =$	→	−3	→	1	→	5	→
$f''(x)$	−	0	+	0	−	0	+

So a possible graph of the original function $y = f(x)$ is



We cannot determine the y-values of the points on the curve but we can be certain of the shape of it. This will be covered in further detail in Chapter 14.

Sketching the reciprocal function

Sometimes we are asked to sketch the graph of a reciprocal function, i.e. $\frac{1}{f(x)}$.

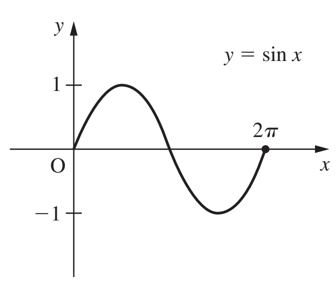
If f(x) is known and a calculator is used, this is easy. However, if it is not known, then we need to consider the following points.

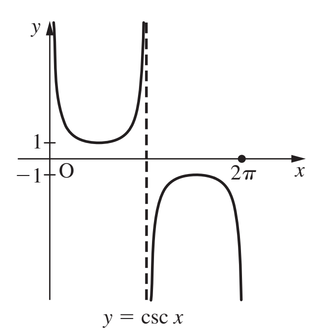
1. At $f(x) = 0$, $\frac{1}{f(x)} \to \infty$. Hence roots on the original graph become vertical asymptotes on the reciprocal graph.
2. At a vertical asymptote, $f(x) \to \infty \Rightarrow \frac{1}{f(x)} = 0$. Hence vertical asymptotes on $f(x)$ become roots on the reciprocal graph.
3. Maximum turning points become minimum turning points and minimum turning points become maximum turning points. The x-value of the turning point stays the same but the y-value is reciprocated.
4. If $f(x)$ is above the x-axis, $\frac{1}{f(x)}$ is also above the x-axis, and if $f(x)$ is below the x-axis, $\frac{1}{f(x)}$ is also below the x-axis.

We will now demonstrate this by example.

Example

If $f(x) = \sin x$, $0 \le x \le 2\pi$ draw $\frac{1}{f(x)}$.

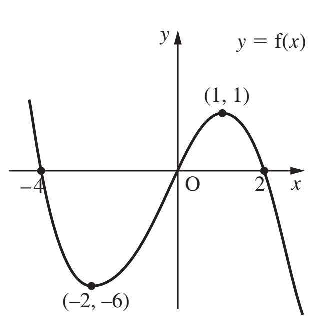


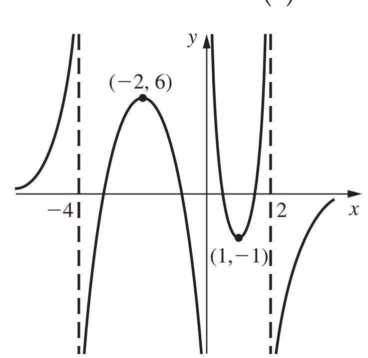


This is the standard way of drawing the curves of $y = \sec x$, $y = \csc x$ and $y = \cot x$.

Example

For the following graph of $y = f(x)$, draw the graph of $y = \frac{1}{f(x)}$.





Exercise 7

1 Sketch the graph of the derived function of the following:

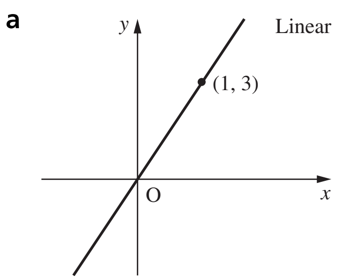
a $y = 4x$ **b** $y = -x$ **c** $y = 4$

d $y = x^2$ **e** $y = -4x^2$ **f** $y = x^2 + x - 7$

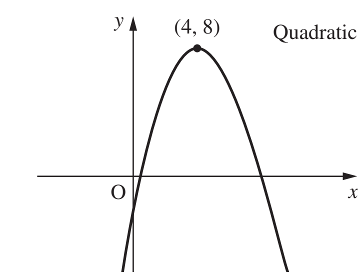
g $y = x^3$ **h** $y = \frac{1}{3}x^3 + 2x^2 + 3x - 8$ **i** $y = \frac{1}{4}x^4$

2 Sketch the graph of the derived function of the following:

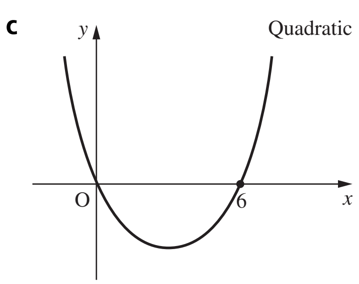
a



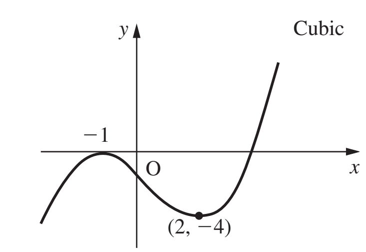
b



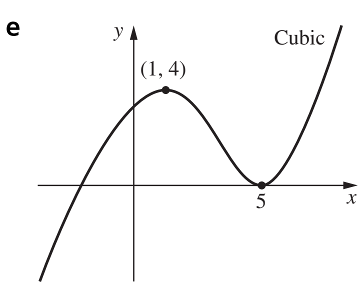
c



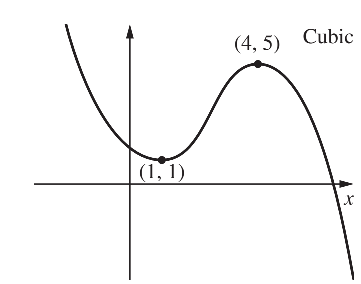
d



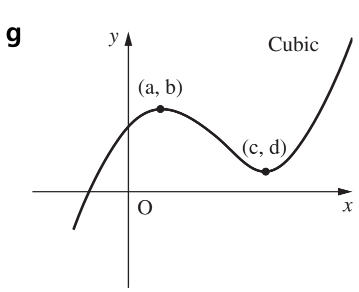
e



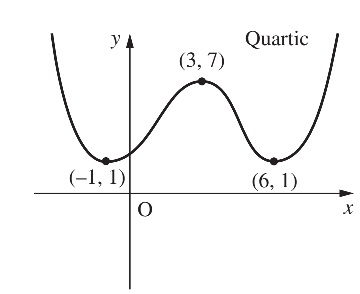
f



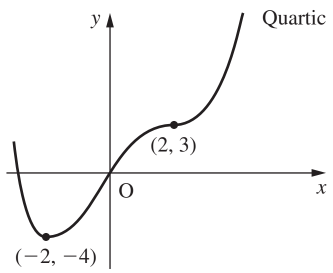
g



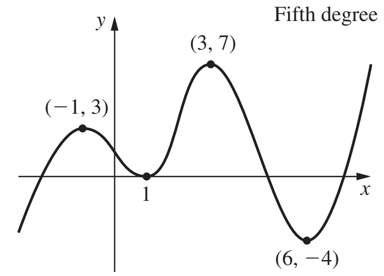
h



i

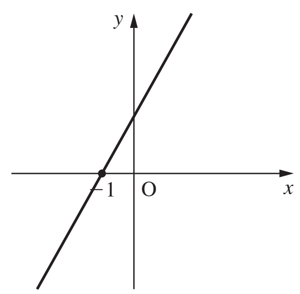


j

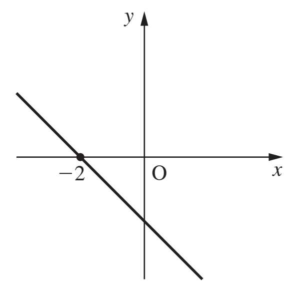


3 Sketch a possible graph of the original function $y = f(x)$, given the derived function graph $y = f'(x)$ in each case.

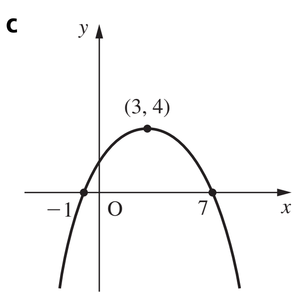
a



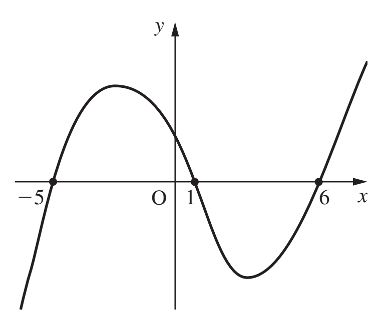
b



c



d



4 For the following functions, draw the graph of $\dfrac{1}{f(x)}$

a $f(x) = 2x - 1$ **b** $f(x) = (x - 2)^2$

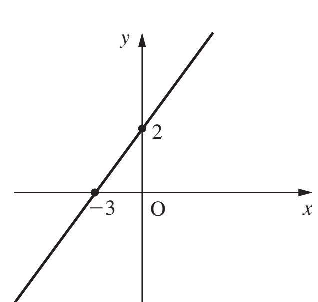
c $f(x) = x^3 - 2x^2 - 5x + 6$ **d** $f(x) = 2x^3 - 15x^2 + 24x + 16$

e $f(x) = e^x$ **f** $f(x) = \ln x$

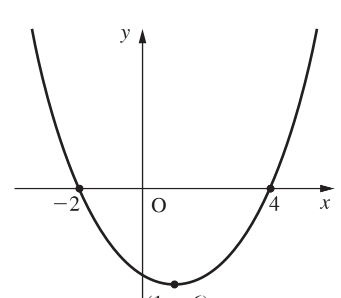
g $f(x) = \cos x,\ 0 \le x \le 2\pi$ **h** $f(x) = \dfrac{6}{(x - 3)}$

5 For the following graphs of $y = f(x)$, sketch the graph of the reciprocal function $y = \dfrac{1}{f(x)}$.

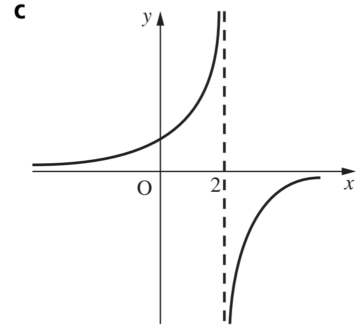
a



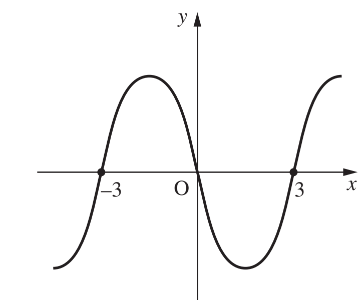
b



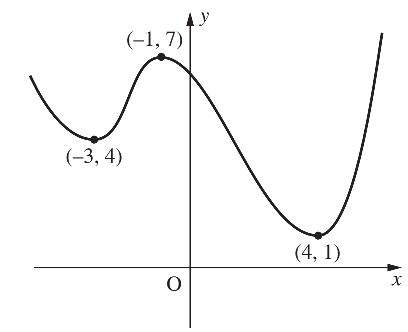
c



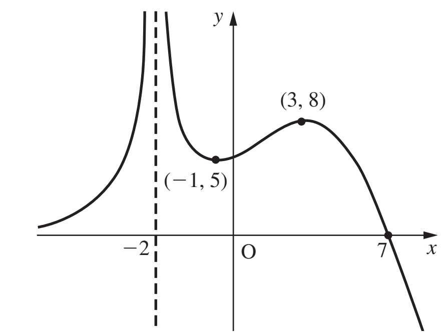
d



e



f



Review exercise

1 Differentiate $f(x) = x^3 - 4x + 5$ using first principles.

2 For $f(x) = \sqrt{x} + \dfrac{2}{x^2}$, find $f'(4)$.

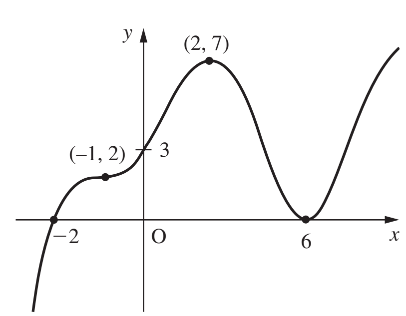
3 Given that $y = \dfrac{3x - x^9}{2\sqrt{x}}$, find $\dfrac{dy}{dx}$.

4 A function is defined as $f(x) = 2x + 3 + \dfrac{64}{x^2}$. Find values of x for which the function is increasing.

5 Given that $f(x) = 5x^2 - 1$ and $g(x) = 3x + 2$, find $h(x) = f(g(x))$. Hence find $h'(x)$.

6 Find the positive value of x for which the gradient of the tangent is -6 for $y = 6x - x^3$. Hence find the equation of the tangent at this point.

7 Sketch the graph of the derivative for the graph below.



8 Find the stationary points of $y = (4x - 1)(2x^2 - 2)$ and investigate their nature.

9 Find the equation of the tangent to $y = x^5 - 3$ at $x = -1$ and find the equation of the normal to $y = 9 - x^2$ at $x = -1$. Find the point where these lines cross.

10 Sketch the graph of $y = \dfrac{x^3}{x^2 - x - 6}$, including all asymptotes, stationary points and intercepts.

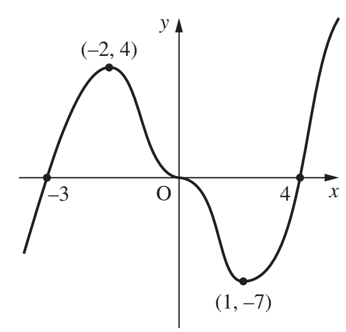
11 Find the equations of all the asymptotes of the graph of $y = \dfrac{x^2 - 5x - 4}{x^2 - 5x + 4}$

[IB Nov 02 p1 Q4]

12 The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point (1, 7). Find the values of a and b. [IB Nov 01 p1 Q7]

13 For the following graphs of $y = f(x)$, draw the graphs of $y = \dfrac{1}{f(x)}$.

a



b