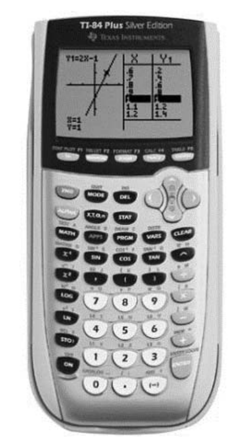
3 Functions

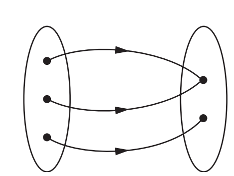
Calculators are an integral part of school mathematics, and this course requires the use of a graphing calculator. Although much of the content of this chapter is "ancient" mathematics in that it has been studied for thousands of years, the use of the calculator in its learning is very recent. Devices for calculating have been around for a long time, the most famous being the abacus. Calculating machines have been built at various times including the difference engine shown below left, built by Charles Babbage in 1822. However, hand-held calculators did not become available until the 1970s, and the first graphing calculator was only produced in 1985. In many ways, the advent of the graphing calculator has transformed the learning of functions and their graphs, and this is a very recent development. What will the next 25 years bring to revolutionise the study of mathematics? Will it be that CAS (computer algebra systems) calculators will become commonplace and an integral part of school mathematics curricula and thus shift the content of these curricula?



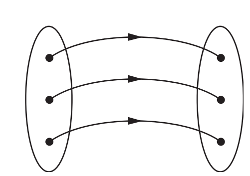
3.1 Functions

A function is a mathematical rule. Although the word "function" is often used for any mathematical rule, this is not strictly correct. For a mathematical rule to be a function, each value of x can have only one image.

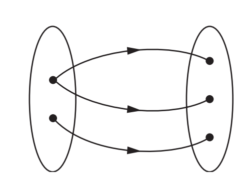
These are arrow diagrams.



Many to one
(a function)



One to one
(a function which has an inverse)



One to many
(not a function)

A simple test can be performed on a graph to find whether it represents a function: if any vertical line cuts more than one point on the graph, it is not a function.

Definitions

Domain – the set of numbers that provide the input for the rule.
Image – the output from the rule of an element in the domain.
Range – the set of numbers consisting of the images of the domain.
Co-domain – a set containing the range.
Function – a rule that links each member of the domain to exactly one member of the range.

This means that a function may have range $y > 0$ but the co-domain could be stated to be $\mathbb{R}$. The term "co-domain" will not be used on examination papers.

Notation

Functions can be expressed in two forms:

$$f: x \rightarrow 2x + 1$$
$$f(x) = 2x + 1$$

The second form is more common but it is important to be aware of both forms.

Finding an image

To find an image, we substitute the value into the function.

Example

Find f(2) for $f(x) = x^3 - 2x^2 + 7x + 1$.

$$\begin{aligned} f(2) &= 2^3 - 2(2)^2 + 7(2) + 1 \\ &= 8 - 8 + 14 + 1 = 15 \end{aligned}$$

Domains and ranges are sets of numbers. It is important to remember the notation of the major sets of numbers.

$\mathbb{Z}$ – the set of integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
$\mathbb{Z}^+$ – the set of positive integers $\{1, 2, 3, \dots\}$
$\mathbb{N}$ – the set of natural numbers $\{0, 1, 2, 3, \dots\}$
$\mathbb{Q}$ – the set of rational numbers $\left\{x : x = \frac{p}{q}, p, q \in \mathbb{Z}\ q \neq 0\right\}$
$\mathbb{R}$ – the set of real numbers.

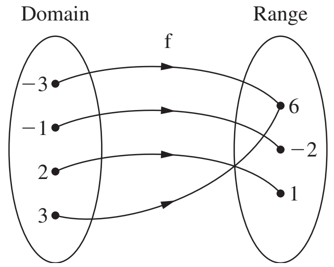
Rational numbers are numbers that can be expressed as a fraction of two integers.

If the domain is not stated, it can be assumed to be the set of real numbers. However, if a domain needs to be restricted for the function to be defined, it should always be given. This is particularly true for rational functions, which are covered later in the chapter.

All numbers on the number line are real numbers (including irrational numbers such as π, $\sqrt{2}$, $\sqrt{3}$).

Example

For $f(t) = t^2 - 3$ with a domain $\{-3, -1, 2, 3\}$, find the range.

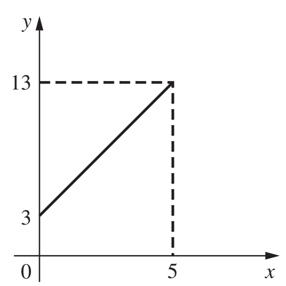


The range is the set of images $= \{-2, 1, 6\}$.

Example

For the function $f(x) = 2x + 3, 0 \le x \le 5$
(a) find f(1)
(b) sketch the graph of f(x)
(c) state the range of this function.

(a) $f(1) = 2(1) + 3 = 5$
(b)



(c) The range is the set of images $= \{3 \le y \le 13\}$.

Example

For $f(x) = x^2 + 5x - 3$, find f(2), f(2x) and $f\left(\dfrac{3}{x}\right)$.

$$\begin{aligned} f(2) &= 2^2 + 5(2) - 3 \\ &= 4 + 10 - 3 \\ &= 11 \end{aligned}$$

To find f(2x), substitute $2x$ for x in the rule f(x).

So $f(2x) = (2x)^2 + 5(2x) - 3$
$= 4x^2 + 10x - 3$

Similarly,

$$\begin{aligned} f\left(\frac{3}{x}\right) &= \left(\frac{3}{x}\right)^2 + 5\left(\frac{3}{x}\right) - 3 \\ &= \frac{9}{x^2} + \frac{15}{x} - 3 \\ &= \frac{9 + 15x - 3x^2}{x^2} \end{aligned}$$

Use an arrow diagram.

Remember, each value of x has only one image in the range.

Exercise 1

1 For the following functions, find f(4).
a $f(x) = 3x - 1$ **b** $f(x) = 9 - 2x$
c $f(x) = x^2 - 3$ **d** $f(x) = \dfrac{24}{x}, x \ne 0$

2 For the following functions, find $g(-2)$.
a $g(t) = 6t - 5$ **b** $g(t) = 3t^2$
c $g(t) = t^3 - 4t^2 + 5t + 7$ **d** $g(t) = \dfrac{17 - t}{t^2}, t \ne 0$

3 Draw an arrow diagram for $f(x) = 4x - 3$ with domain $\{-1, 1, 5\}$ and state the range.

4 Draw an arrow diagram for $g(x) = \dfrac{x - 3}{x}$ with domain $\{-3, -1, 1, 6\}$ and state the range.

5 Draw the graph of $f(x) = x - 5$ for $0 \le x \le 7$ and state the range.

6 Draw the graph of $f(x) = 9 - 2x$ for $-3 \le x \le 2$ and state the range.

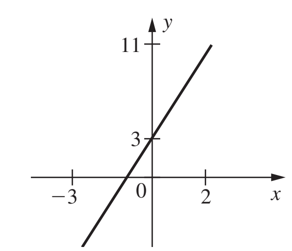
7 Draw the graph of $g(x) = x^2 - x - 6$ for $0 \le x \le 6$ and state the range.

8 Draw the graph of $p(x) = 2x^3 - 7x + 6$ for $-1 \le x \le 5$ and state the range.

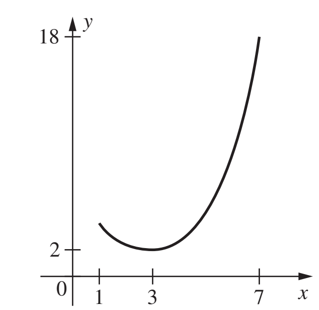
9 For $f(x) = 2x + 5, x \in \mathbb{R}^+$, what is the range?

10 For each of these graphs, state the domain and range.

a



b



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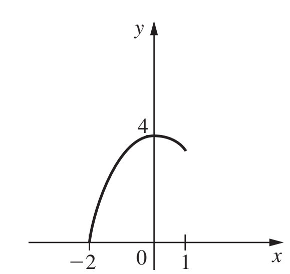


11 For $f(x) = 3x - 2$, find
a f(2x) **b** $f(-x)$ **c** $f\left(\dfrac{1}{x}\right)$

12 For $g(x) = x^2 - 3x$, find
a g(2x) **b** $g(x + 4)$ **c** g(6x) **d** $g(2x - 1)$

13 For $h(x) = \dfrac{x}{2 - x}$, find
a $h(-x)$ **b** h(4x) **c** $h\left(\dfrac{1}{x}\right)$ **d** $h(x + 2)$

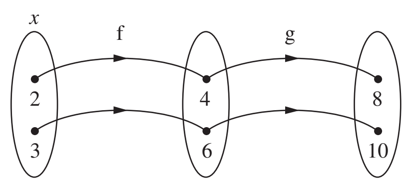
14 For $k(x) = x - 9$, find $k(x + 9)$.

3.2 Composite functions

When one function is followed by another, the resultant effect can be expressed as a single function. When functions are combined like this, the resultant function is known as a **composite function**.

Example

Find $h(x) = g(f(x))$ if
$f(x) = 2x$
$g(x) = x + 4$



This composite function is f followed by g.
So in this example the effect is $\times 2$ then $+4$.
$h(x) = 2x + 4$

g(f(x)) is f followed by g (the order is important).

This was a very simple example, but for more complicated functions it is useful to find the composite function in two steps like this:

$g(f(x)) = g(2x)$ (substituting $2x$ for x in $g(x)$)
$= 2x + 4$

f(g(x)) is sometimes also written $(f \circ g)(x)$

Example

For $f(x) = 2x - 1$ and $g(x) = x^2 + 5x - 2$, find $g(f(x))$.

$$\begin{aligned} g(f(x)) &= g(2x - 1) \\ &= (2x - 1)^2 + 5(2x - 1) - 2 \\ &= 4x^2 - 4x + 1 + 10x - 5 - 2 \\ &= 4x^2 + 6x - 6 \end{aligned}$$

It is important to note that the order of the functions is very important.

In general, $g(f(x)) \neq f(g(x))$.

In the example
$$\begin{aligned} f(g(x)) &= f(x^2 + 5x - 2) \\ &= 2(x^2 + 5x - 2) - 1 \\ &= 2x^2 + 10x - 4 - 1 \\ &= 2x^2 + 10x - 5 \end{aligned}$$

Example

For $h(\theta) = \sec\theta$ and $p(\theta) = \sin 2\theta - 4\theta$, find $p(h(\theta))$.

$$\begin{aligned} p(h(\theta)) &= p(\sec\theta) \\ &= \sin(2\sec\theta) - 4\sec\theta \end{aligned}$$

Example

For $f(x) = \dfrac{2}{x-1}$, $x \neq 1$, and $g(x) = x + 4$, find (a) $g(f(4))$ (b) $f(g(x))$

(a) $f(4) = \dfrac{2}{4-1} = \dfrac{2}{3}$

$$\begin{aligned} g(f(4)) &= g\left(\frac{2}{3}\right) \\ &= \frac{2}{3} + 4 \\ &= \frac{14}{3} \end{aligned}$$

(b) $$\begin{aligned} f(g(x)) &= f(x + 4) \\ &= \frac{2}{x + 4 - 1} \\ &= \frac{2}{x + 3} \end{aligned}$$

Note that we did not find g(f(x)) here in order to find g(f(4)). It is generally easier to just calculate f(4) and then input this value into g(x).

Exercise 2

1 For $f(x) = x + 6$ and $g(x) = 4x$, find
- **a** $f(g(2))$
- **b** $g(f(0))$
- **c** $f(g(-1))$
- **d** $g(f(x))$

2 For $h(x) = 3x + 2$ and $p(x) = x^2 - 3$, find
- **a** $p(h(2))$
- **b** $h(p(-2))$
- **c** $p(h(-3))$
- **d** $h(p(x))$

3 For $f(\theta) = \sin\theta$ and $g(\theta) = \theta + \dfrac{\pi}{3}$, find
- **a** $g\left(f\left(\dfrac{\pi}{2}\right)\right)$
- **b** $f\left(g\left(\dfrac{\pi}{3}\right)\right)$
- **c** $f(g(-2\pi))$
- **d** $f(g(\theta))$

4 For each pair of functions, find (i) f(g(x)) and (ii) g(f(x)).
- **a** $f(x) = 2x - 1$, $g(x) = x^2$
- **b** $f(x) = x^2 - 4$, $g(x) = 3x + 5$
- **c** $f(x) = x^3$, $g(x) = x^2 - 6$
- **d** $f(x) = \cos x$, $g(x) = 3x^2$
- **e** $f(x) = x^3 - x + 7$, $g(x) = 2x + 3$
- **f** $f(x) = x^6 - 2x + 3$, $g(x) = x^2 + 4$
- **g** $f(x) = \sin 2x$, $g(x) = x^2 - 7$

5 For $f(x) = 3x + 4$ and $g(x) = 2x - p$, where p is a constant, find

a $f(g(x))$ **b** $g(f(x))$ **c** p if $f(g(x)) = g(f(x))$

6 For $f(x) = 6x^2$ and $g(x) = 2x - 3$, find

a $g(f(x))$ **b** $f(g(x))$ **c** $f(f(x))$ **d** $g(g(x))$

7 For $f(x) = x + \frac{\pi}{2}$ and $g(x) = \cos x$, find

a $g(f(x))$ **b** $f(g(x))$ **c** $f(f(x))$ **d** $g(g(x))$

8 For each pair of functions, find (i) $f(g(x))$ and (ii) $g(f(x))$, in simplest form.

a $f(x) = \frac{2}{x-3}, x \neq 3, g(x) = 3x + 1$

b $f(x) = x^2 - 3x, g(x) = \frac{3}{x}, x \neq 0$

c $f(x) = 2 - 5x, g(x) = \frac{x}{x+1}, x \neq -1$

d $f(x) = \frac{2}{3x-1}, x \neq \frac{1}{3}, g(x) = \frac{1}{x}, x \neq 0$

e $f(x) = \frac{x}{2+x}, x \neq -2, g(x) = \frac{2}{x}, x \neq 0$

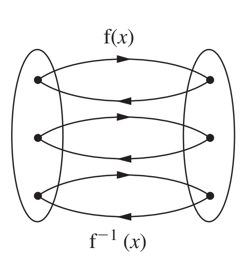
9 For $f(x) = \frac{1}{x+7}, x \neq -7$, and $g(x) = \frac{1}{x} - 7, x \neq 0$, find $f(g(x))$ in its simplest form.

3.3 Inverse functions

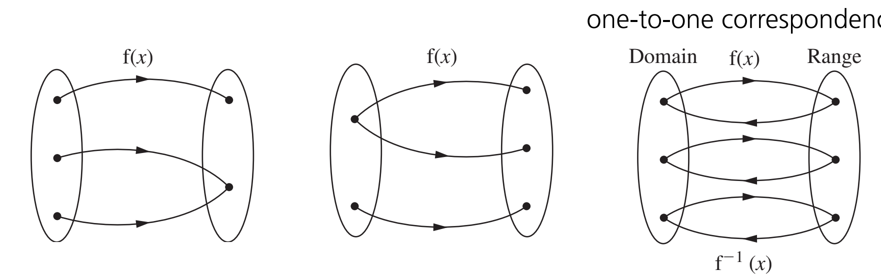
Consider a function $y = f(x)$. If we put in a single value of x, we find a single value of y. If we were given a single value of y and asked to find the single value of x, how would we do this? This is similar to solving a linear equation for a linear function. The function that allows us to find the value of x is called the **inverse function**. Note that the original function and its inverse function are inverses of each other. We know that addition and subtraction are opposite operations and division is the opposite of multiplication. So, the inverse function of $f(x) = x + 3$ is $f^{-1}(x) = x - 3$.

$f^{-1}(x)$ is the inverse of $f(x)$.

Note that both the original function $f(x)$ and the inverse function $f^{-1}(x)$ are functions. This means that for both functions there can only be one image for each element in the domain. This can be illustrated in an arrow diagram.



This means that not all functions have an inverse. An inverse exists only if there is a one-to-one correspondence between domain and range in the function.



No inverse exists for the function $f(x)$

$f(x)$ is not a function

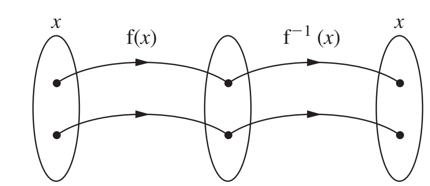
$f(x)$ is a function and an inverse function exists

An inverse does not exist for a many-to-one function, unless the domain is restricted.

When testing whether a mapping was a function, we used a vertical line test. To test if a mapping is a one-to-one correspondence, we can use a vertical and horizontal line test. If the graph crosses any vertical line more than once, the graph is not a function. If the graph crosses any horizontal line more than once, there is no inverse (for that domain).

The arrow diagram for a one-to-one correspondence shows that the range for $f(x)$ becomes the domain for $f^{-1}(x)$. Also, the domain of $f(x)$ becomes the range for $f^{-1}(x)$.

Looking at $f(x)$ and $f^{-1}(x)$ from a composite function view, we have



that is $f^{-1}(f(x)) = x$.

Finding an inverse function

For some functions, the inverse function is obvious, as in the example of $f(x) = x + 3$, which has inverse function $f^{-1}(x) = x - 3$. However, for most functions more thought is required.

Method for finding an inverse function

1. Check that an inverse function exists for the given domain.
2. Rearrange the function so that the subject is x.
3. Interchange x and y.

Example

Find the inverse function for $f(x) = 2x - 1$.

Here there is no domain stated and so we assume that it is for $x \in \mathbb{R}$. $f(x) = 2x - 1$ has a one-to-one correspondence for all real numbers and so an inverse exists.

Let $y = 2x - 1$

So $2x = y + 1$

$\Rightarrow x = \frac{1}{2}y + \frac{1}{2}$

Interchanging x and y gives $y = \frac{1}{2}x + \frac{1}{2}$.

So the inverse function is $f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$.

Example

Find the inverse function for $f(x) = \frac{6}{x-2}$ for $x > 2, x \in \mathbb{R}$.

An inverse exists as there is a one-to-one correspondence for f(x) when $x > 2$.

Let $y = \frac{6}{x-2}$

So $x - 2 = \frac{6}{y}$

$\Rightarrow x = \frac{6}{y} + 2$

$\Rightarrow x = \frac{2y+6}{y}$

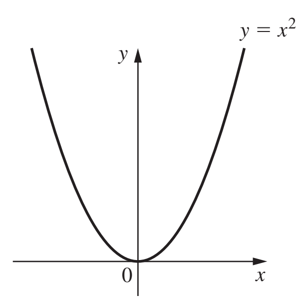
Interchanging x and y gives $y = \frac{2x+6}{x}$.

So the inverse function is $f^{-1}(x) = \frac{2x+6}{x}, x \in \mathbb{R}^+$.

Example

For $f(x) = x^2$, find $f^{-1}(x)$ for a suitable domain.

Considering the graph of $f(x)$, it is clear that there is not a one-to-one correspondence for all $x \in \mathbb{R}$.



However, if we consider only one half of the graph and restrict the domain to $x \in \mathbb{R}, x \geq 0$, an inverse does exist.

Let $y = x^2$

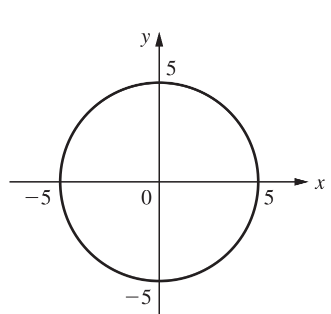
Then $x = \sqrt{y}$

Interchanging x and y gives $y = \sqrt{x}$ (positive root only)

$\Rightarrow f^{-1}(x) = \sqrt{x}, x \in \mathbb{R}, x \geq 0$

This is important to note: a square root is only a function if only the positive root is considered.

The equation $x^2 + y^2 = 25$ is for a circle with centre the origin, as shown below.



This graph is not a function as there exist vertical lines that cut the graph twice. There are also horizontal lines that cut the circle twice. Even restricting the domain to $0 \leq x \leq 5$ does not allow there to be an inverse as the original graph is not a function. So it is not possible to find an inverse for $x^2 + y^2 = 25$.

Exercise 3

1 Which of the following have an inverse function for $x \in \mathbb{R}$?

a $f(x) = 2x - 3$ **b** $f(x) = x^2 - 1$ **c** $f(x) = x^3$ **d** $y = \cos x$

2 For each function f(x), find the inverse function $f^{-1}(x)$.

a $f(x) = 4x$ **b** $f(x) = x - 5$ **c** $f(x) = x + 6$

d $f(x) = \frac{2}{3}x$ **e** $f(x) = 7 - x$ **f** $f(x) = 9 - 4x$

g $f(x) = 2x + 9$ **h** $f(x) = x^3 - 6$ **i** $f(x) = 8x^3$

3 What is the largest domain for which f(x) has an inverse function?

a $f(x) = \frac{1}{x-3}$ **b** $f(x) = \frac{2}{x+4}$ **c** $f(x) = \frac{3}{2x-1}$

d $f(x) = x^2 - 5$ **e** $f(x) = 9 - x^2$ **f** $f(x) = x^2 - x - 12$

g $f(x) = \cos x$

4 For each function f(x), (i) choose a suitable domain so that an inverse exists (ii) find the inverse function $f^{-1}(x)$.

a $f(x) = \frac{1}{x-6}$ **b** $f(x) = \frac{3}{x+7}$ **c** $f(x) = \frac{5}{3x-2}$

d $f(x) = \frac{7}{2-x}$ **e** $f(x) = \frac{8}{4x-9}$ **f** $f(x) = \frac{4}{5x+6}$

g $f(x) = 6x^2$ **h** $f(x) = x^2 - 4$ **i** $f(x) = 2x^2 + 3$

j $f(x) = 16 - 9x^2$ **k** $f(x) = x^4$ **l** $f(x) = 2x^3 - 5$

5 For $f(x) = 3x$ and $g(x) = x - 2$, find

a $h(x) = f(g(x))$ **b** $h^{-1}(x)$

3.4 Graphs of inverse functions

On a graphing calculator, graph the following functions and their inverse functions and look for a pattern:

1 $f(x) = 2x, \ f^{-1}(x) = \frac{1}{2}x$

2 $f(x) = x + 4, \ f^{-1}(x) = x - 4$

3 $f(x) = x - 1, \ f^{-1}(x) = x + 1$

4 $f(x) = 3x - 1$, $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$

5 $f(x) = \frac{2}{x - 3}$, $f^{-1}(x) = \frac{2 + 3x}{x}$ for $x > 3$

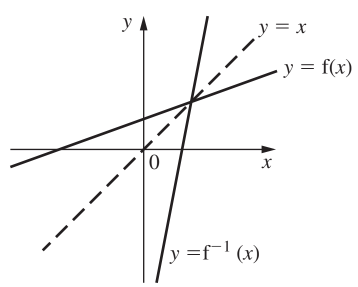
Through this investigation it should be clear that the graph of a function and its inverse are connected. The connection is that one graph is the reflection of the other in the line $y = x$.

This connection should make sense. By reflecting the point (x, y) in the line $y = x$, the image is (y, x). In other words, the domain becomes the range and vice versa. This reflection also makes sense when we remember that $f(f^{-1}(x)) = x$.

Thus if we have a graph (without knowing the equation), we can sketch the graph of the inverse function.

Example

For the graph of $f(x)$ below, sketch the graph of its inverse, $f^{-1}(x)$.



Drawing in the line $y = x$ and then reflecting the graph in this line produces the graph of the inverse function $f^{-1}(x)$.

Exercise 4

1 For each function $f(x)$, find the inverse function $f^{-1}(x)$ and draw the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram.

a $f(x) = 2x$ **b** $f(x) = x + 2$ **c** $f(x) = x - 3$

d $f(x) = 3x + 1$ **e** $f(x) = 2x - 4$

2 For each function $f(x)$, draw the graph of $y = f(x)$ for $x \geq 0$. Find the inverse function $f^{-1}(x)$ and draw it on the same graph.

a $f(x) = x^2$ **b** $f(x) = 3x^2$

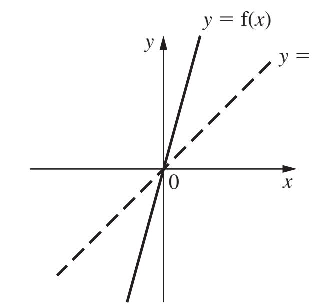
c $f(x) = x^2 + 4$ **d** $f(x) = 5 - x^2$

3 For each function $f(x)$, draw the graph of $y = f(x)$ for $x \geq 0$. Find the inverse function $f^{-1}(x)$ and draw it on the same graph.

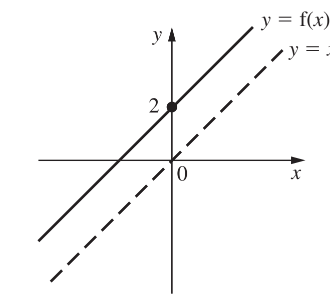
a $f(x) = \frac{1}{x + 2}$ **b** $f(x) = \frac{1}{x + 5}$ **c** $f(x) = \frac{2}{x + 1}$

4 Sketch the graph of the inverse function $f^{-1}(x)$ for each graph.

a



b



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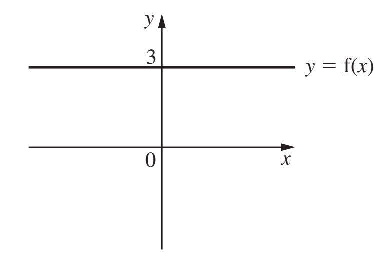

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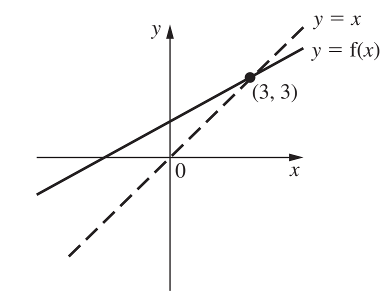
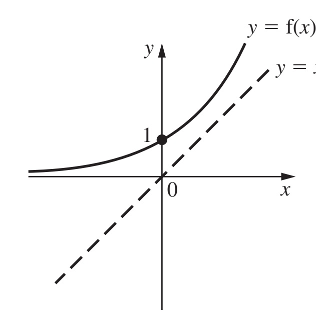
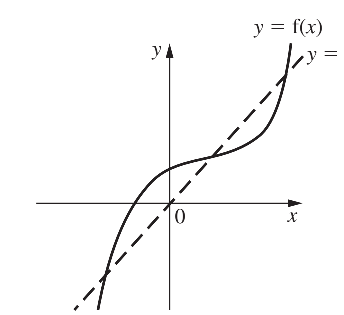


e



f



3.5 Special functions

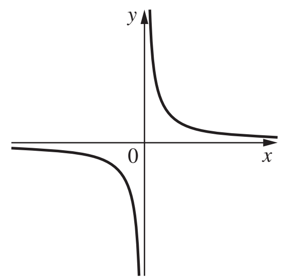
The reciprocal function

The function known as the reciprocal function is $f(x) = \frac{1}{x}$. In Chapter 1, we met vertical asymptotes. These occur when a function is not defined (when the denominator is zero). For $f(x) = \frac{1}{x}$, there is a vertical asymptote when $x = 0$. To draw the graph, we consider what happens either side of the asymptote. When $x = -0.1$, $f(-0.1) = -10$. When $x = 0.1$, $f(0.1) = 10$.

Now consider what happens for large values of x, that is, as $x \to \pm\infty$.

As $x \to \infty$, $\frac{1}{x} \to 0$. As $x \to -\infty$, $\frac{1}{x} \to 0$.

So the graph of $f(x) = \frac{1}{x}$ is



For $x \to \infty, y \to 0$. For large values of x, the graph approaches the x-axis. This is known as a horizontal asymptote. As with vertical asymptotes, this is a line that the graph approaches but does not reach.

It is clear from the graph that this function has an inverse, provided $x \neq 0$.

Let $y = \frac{1}{x}$

$\Rightarrow x = \frac{1}{y}$

Interchanging y and x, $y = \frac{1}{x}$.

Hence $f^{-1}(x) = \frac{1}{x}, x \neq 0, x \in \mathbb{R}$. So this function is the inverse of itself. Hence it has a self-inverse nature and this is an important feature of this function.

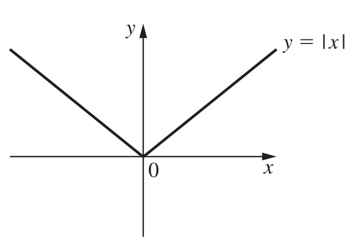
The absolute value function

The function denoted $f(x) = |x|$ is known as the **absolute value function**. This function can be described as making every y value positive, that is, ignoring the negative sign. This can be defined strictly as

$$f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

This is known as a **piecewise function** as it is defined in two pieces.

This is the graph:

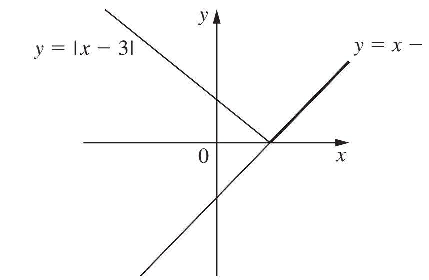


The whole graph is contained above the x-axis. Note the unusual "sharp" corner at $x = 0$.

The absolute value can be applied to any function. The effect is to reflect in the x-axis any part of the graph that is below the x-axis while not changing any part above the x-axis.

Example

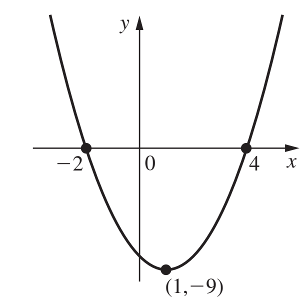
Sketch the graph of $y = |x - 3|$.



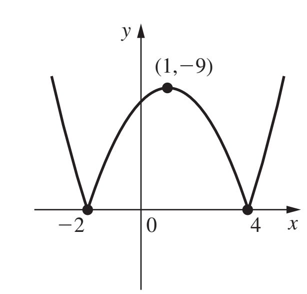
Example

Sketch the graph of $y = |x^2 - 2x - 8|$.

Start by sketching the graph of $y = x^2 - 2x - 8$
$= (x - 4)(x + 2)$



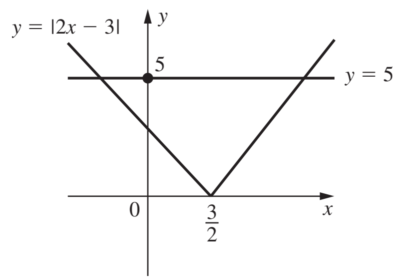
Reflect the negative part of the graph in the x-axis:



The graph of an absolute value function can be used to solve an equation or inequality.

Example

Solve $|2x - 3| = 5$.



For the negative solution solve

$-(2x - 3) = 5$
$\Rightarrow -2x + 3 = 5$
$\Rightarrow -2x = 2$
$\Rightarrow x = -1$

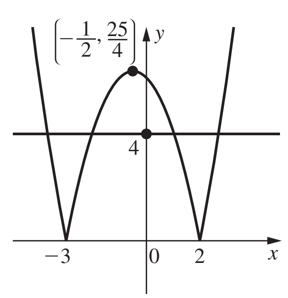
For the positive solution solve

$2x - 3 = 5$
$\Rightarrow 2x = 8$
$\Rightarrow x = 4$

Example

Solve $|x^2 + x - 6| \le 4$.

$x^2 + x - 6 = (x + 3)(x - 2)$ so the graph of $y = |x^2 + x - 6|$ is



To find the points of intersection of $y = |x^2 + x - 6|$ and $y = 4$ solve

$-x^2 - x + 6 = 4$
$\Rightarrow x^2 + x - 2 = 0$
$\Rightarrow (x + 2)(x - 1) = 0$
$\Rightarrow x = -2$ or $x = 1$

and

$x^2 + x - 6 = 4$
$\Rightarrow x^2 + x - 10 = 0$
$\Rightarrow x = -3.70$ or $x = 2.70$
(using the quadratic formula)

Hence $|x^2 + x - 6| \le 4 \Rightarrow -3.70 \le x \le -2$ or $1 \le x \le 2.70$

Exercise 5

1 Write $f(x) = |x - 2|$ as a piecewise function.
2 Write $f(x) = |2x + 1|$ as a piecewise function.
3 Write $f(x) = |x^2 - x - 12|$ as a piecewise function.
4 Write $f(x) = |2x^2 - 5x - 3|$ as a piecewise function.

5 Sketch the graph of $y = |x + 4|$.
6 Sketch the graph of $y = |3x|$.
7 Sketch the graph of $y = |3x - 5|$.
8 Sketch the graph of $y = |x^2 + 4x - 12|$.
9 Sketch the graph of $y = |x^2 - 7x + 12|$.
10 Sketch the graph of $y = |x^2 + 5x + 6|$.
11 Sketch the graph of $y = |3x^2 + 5x - 2|$.
12 Solve $|x + 2| = 3$.
13 Solve $|x - 5| = 1$.
14 Solve $|2x + 5| = 3$.
15 Solve $|7 - 2x| = 3$.
16 Solve $|x^2 + x - 6| = 2$.
17 Solve $|2x^2 + x - 10| = 4$.
18 Solve $|x + 2| < 5$.
19 Solve $|2x - 1| \le 9$.
20 Solve $|9 - 4x| < 1$.
21 Solve $|x^2 + 4x - 12| \le 7$.
22 Solve $|2x^2 + 5x - 12| < 9$.

3.6 Drawing a graph

In the first two chapters, we covered drawing trigonometric graphs and drawing quadratic graphs. We have now met some of the major features of the graphs, including

- roots – values of x when $y = 0$
- y-intercept – the value of y when $x = 0$
- turning points
- vertical asymptotes – when y is not defined
- horizontal asymptotes – when $x \to \pm\infty$

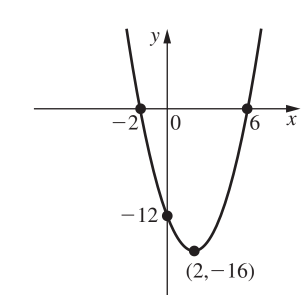
More work will be done on sketching graphs in Chapter 8.

Example

Sketch the graph of $y = x^2 - 4x - 12$, noting the major features.

This process was covered in Chapter 2.

$x^2 - 4x - 12 = (x + 2)(x - 6)$
so the graph has roots at $x = -2$ and $x = 6$.
We know the shape of this function, and that it has a minimum turning point at $(2, -16)$ by the symmetry of the graph.
Setting $x = 0$ gives the y-intercept as $y = -12$.
This graph has no asymptotes.



Example

Sketch the graph of $y = \dfrac{2}{x - 3}$.

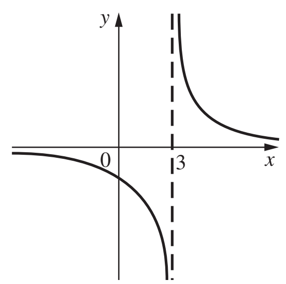
This graph has no roots as the numerator is never zero.

When $x = 0$, $y = -\dfrac{2}{3}$ so $\left(0, -\dfrac{2}{3}\right)$ is the y-intercept.

There is a vertical asymptote when $x - 3 = 0 \Rightarrow x = 3$.

As $x \to \pm\infty$, the denominator becomes very large and so $y \to 0$.

By taking values of x close to the vertical asymptote, we can determine the behaviour of the graph around the asymptote. So, when $x = 2.9$, $y = -20$. When $x = 3.1$, $y = 20$. So the graph is

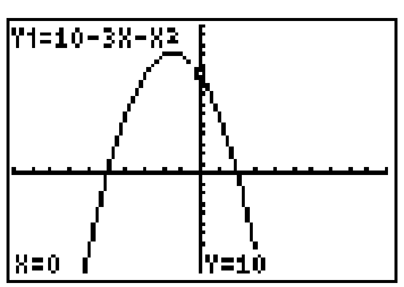


This is an example of a rational function. More work on these is covered on page 79.

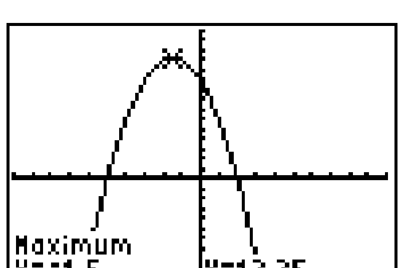
All of these features can be identified when sketching a function using a graphing calculator.

Example

Sketch the graph of $y = 10 - 3x - x^2$.



The calculator can be used to calculate points such as intercepts and turning points. The asymptotes (if any) are clear from the graph (as long as an appropriate window is chosen).



$x = -5$ and $x = 2$ are the roots.

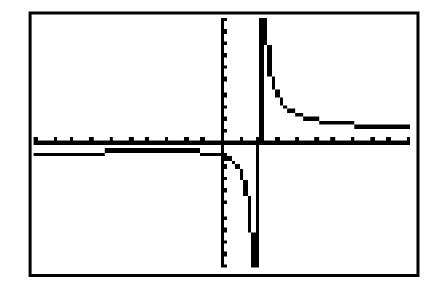
$(0, 10)$ is the y-intercept.

$\left(-\dfrac{3}{2}, \dfrac{49}{4}\right)$ is the maximum turning point.

Many types of function can be sketched using a graphing calculator. Although we study a number of functions in detail, including straight lines, polynomials and trigonometric functions, there are some functions that we only sketch using the calculator (in this course).

Example

Sketch the graph of $y = \dfrac{(x^2 + 5)^{\frac{1}{2}}}{x - 2}$.



Here we can see that there is a vertical asymptote at $x = 2$, a horizontal asymptote at $y = 0$, there is no turning point, and the y-intercept is $-\dfrac{\sqrt{5}}{2}$.

Exercise 6

1 For each function, sketch the graph of $y = f(x)$, indicating asymptotes, roots, y-intercepts and turning points.

a $f(x) = x + 4$ **b** $f(x) = 2x - 1$

c $f(x) = 3 - x$ **d** $f(x) = 7 - 2x$

e $f(x) = x^2 + 7x + 12$ **f** $f(x) = x^2 - 8x + 12$

g $f(x) = x^2 - 5x - 24$ **h** $f(x) = 3x^2 + 2x - 8$

i $f(x) = 6x^2 + x - 15$ **j** $f(x) = 20 + 17x - 10x^2$

k $f(x) = \dfrac{1}{x + 4}$ **l** $f(x) = \dfrac{3}{x - 2}$

m $f(x) = \dfrac{4}{2x - 1}$

2 Using a graphing calculator, make a sketch of $y = f(x)$, indicating asymptotes, roots, y-intercepts and turning points.

a $f(x) = x^2 - x - 30$ **b** $f(x) = x^2 + 5x + 3$

c $f(x) = x^2 + 2x + 5$ **d** $f(x) = \dfrac{6}{2x + 3}$

e $f(x) = \dfrac{5}{(x + 2)(x - 3)}$ **f** $f(x) = \dfrac{7}{x^2 - 7x + 12}$

g $f(x) = \dfrac{x + 1}{x + 5}$ **h** $f(x) = \dfrac{x + 3}{x^2 - 3x - 10}$

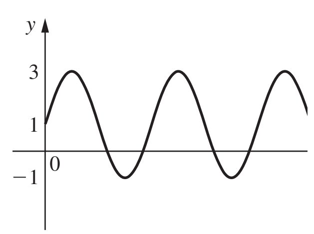
i $f(x) = \dfrac{x - 2}{x^2 + 6}$ **j** $f(x) = \dfrac{x^2 + 8}{x - 5}$

k $f(x) = \dfrac{x^2 - x - 6}{x^2 + 10x + 24}$ **l** $f(x) = \dfrac{(x^2 + 3)^3(x + 2)}{x + 7}$

m $f(x) = \dfrac{\sin x}{x^2}$ **n** $f(x) = \dfrac{\cos x^2}{x + 1}$

3.7 Transformations of functions

In Chapter 1, we met trigonometric graphs such as $y = 2 \sin 3x + 1$.

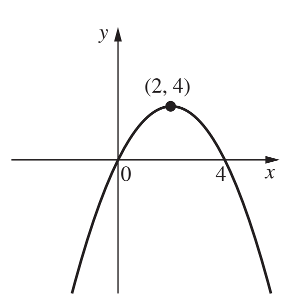


Here the 3 has the effect of producing three waves in 2π (three times as many graphs as $y = \sin x$).

The 2 stretches the graph vertically.

The 1 shifts the graph vertically.

In Chapter 2, we met quadratic graphs such as $y = -(x - 2)^2 + 4$.



Here the -2 inside the bracket has the effect of shifting $y = x^2$ right.

The -1 in front of the bracket reflects the graph in the x-axis.

The $+4$ shifts the graph vertically.

We can see that there are similar effects for both quadratic and trigonometric graphs. We can now generalize transformations as follows:

For $kf(x)$, each y-value is multiplied by k and so this creates a vertical stretch.

For $f(kx)$, each x-value is multiplied by k and so this creates a horizontal stretch.

For $f(x) + k$, k is added to each y-value and so the graph is shifted vertically.

For $f(x + k)$, k is added to each x-value and so the graph is shifted horizontally.

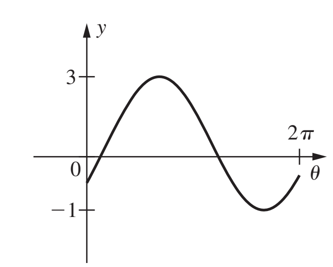
For $-f(x)$, each y-value is multiplied by -1 and so each point is reflected in the x-axis.

For $f(-x)$, each x-value is multiplied by -1 and so each point in reflected in the y-axis.

General form	Example	Effect
$kf(x)$	$y = 3 \sin x$	Vertical stretch
$f(kx)$	$y = \cos 2x$	Horizontal stretch
$f(x) + k$	$y = x^2 + 5$	Vertical shift [$k > 0$ up, $k < 0$ down]
$f(x + k)$	$y = (x + 3)^2$	Horizontal shift [$k > 0$ left, $k < 0$ right]
$-f(x)$	$y = -\cos x$	Reflection in x-axis
$f(-x)$	$y = \sin(-x)$	Reflection in y-axis

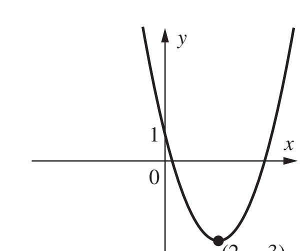
Example

Sketch the graph of $y = -2 \cos\left(\theta + \dfrac{\pi}{4}\right) + 1$.



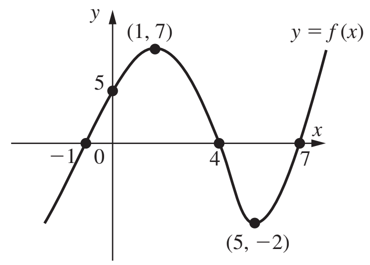
Example

Sketch the graph of $y = (x - 2)^2 - 3$.

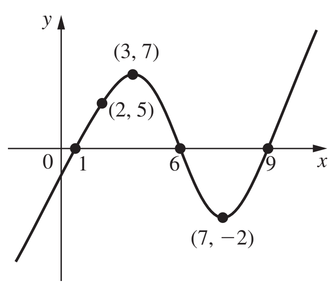


Example

This is a graph of $y = f(x)$.
Draw
(a) $f(x - 2)$
(b) $f(-x)$

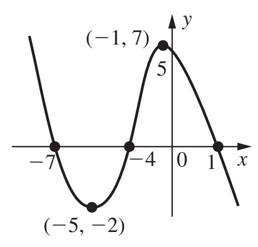


(a) $f(x - 2)$



This is a horizontal shift of 2 to the right.

(b) $f(-x)$

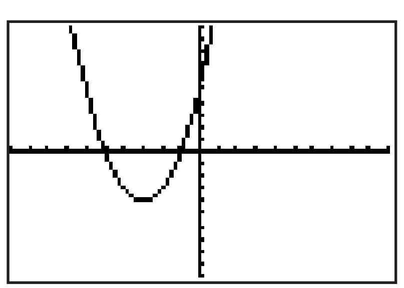


This is a reflection in the x-axis.

The absolute value of a function $|f(x)|$ can be considered to be a transformation of a function (one that reflects any parts below the x-axis). A graphing calculator can also be used to sketch transformations of functions, as shown below.

Example

Given that $f(x) = x^2 - 4$, sketch the graph of (a) $f(x + 3)$ (b) $|f(x)|$
(a)

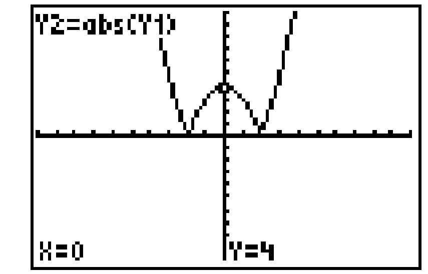


As expected, this is the graph of $f(x)$ shifted 3 places to the left. The calculator cannot find the "new" function, but the answer can be checked once it is found algebraically, if required.

$$\begin{aligned} f(x + 3) &= (x + 3)^2 - 4 \\ &= x^2 + 6x + 9 - 4 \\ &= x^2 + 6x + 5 \end{aligned}$$

This can be sketched and checked to be the same as $f(x + 3)$.

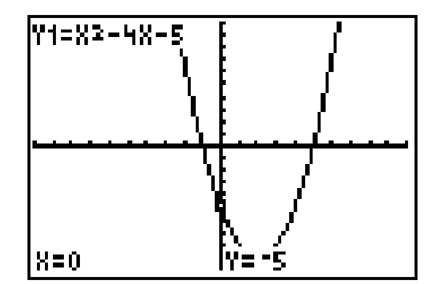
(b) The negative part of the curve is reflected in the x-axis, that is, the part defined by $-2 \le x \le 2$.

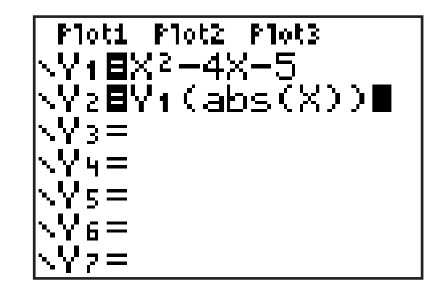


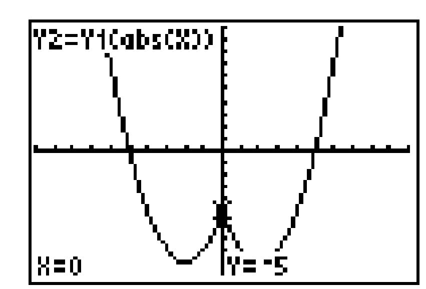
Similarly, the function of the absolute value of x, that is, $f(|x|)$, can be considered to be a transformation of a function. If we consider this as a piecewise function, we know that the absolute value part will have no effect for $x \ge 0$. However, for $x < 0$, the effect will be that it becomes $f(-x)$. This means that the graph of $f(|x|)$ will be the graph of $f(x)$ for $x \ge 0$ and this will then be reflected in the y-axis.

Example

Sketch the graph of $f(x) = x^2 - 4x - 5$ and the graph of $f(|x|)$.
Using a graphing calculator, we can sketch both graphs:







Rational functions

Rational functions are functions of the type $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials. Here we shall consider functions of the type $\dfrac{a}{px + q}$ and $\dfrac{bx + c}{px + q}$.

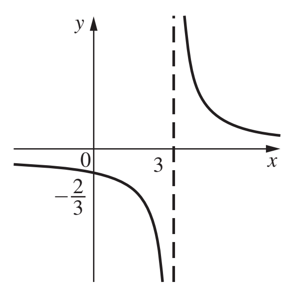
Functions of the type $\dfrac{a}{px+q}$

These can be considered to be a transformation of the reciprocal function $f(x) = \dfrac{1}{x}$.

Example

Sketch the graph of $y = \dfrac{2}{x-3}$.

Comparing this to $f(x) = \dfrac{1}{x}$, $y = 2f(x-3)$. So its graph is the graph of $y = \dfrac{1}{x}$, stretched vertically $\times 2$ and shifted 3 to the right.

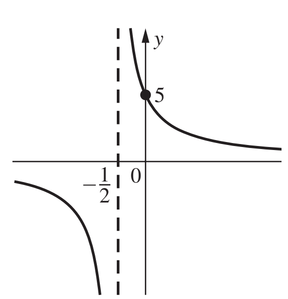


Example

Sketch the graph of $y = \dfrac{5}{2x+1}$.

To consider this as a transformation, it can be written as $y = \dfrac{\frac{5}{2}}{x+\frac{1}{2}}$.

This is $y = \dfrac{1}{x}$ shifted left $\dfrac{1}{2}$ and vertically stretched by $\dfrac{5}{2}$. However, it is probably easiest just to calculate the vertical asymptote and the y-intercept. Here the vertical asymptote is when $2x + 1 = 0 \Rightarrow x = -\dfrac{1}{2}$. The y-intercept is when $x = 0 \Rightarrow y = 5$.



Functions of the type $\dfrac{bx+c}{px+q}$

The shape of this graph is very similar to the previous type but the horizontal asymptote is not the x-axis.

The horizontal asymptote is $y = \dfrac{b}{p}$, as when $x \to \pm\infty$, $y \to \dfrac{b}{p}$.

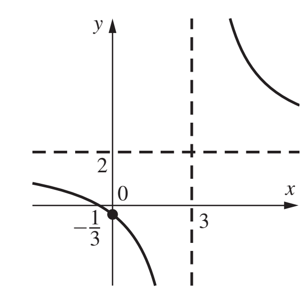
Example

Sketch the graph of $y = \dfrac{2x+1}{x-3}$.

This has a vertical asymptote at $x = 3$.

The horizontal asymptote is $y = 2$. [As $x \to \pm\infty$, $y \to \dfrac{2x}{x} = 2$]

The y intercept is $y = -\dfrac{1}{3}$.



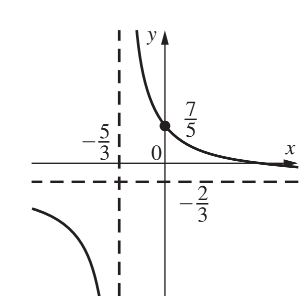
Example

Sketch the graph of $y = \dfrac{7-2x}{3x+5}$.

This has a vertical asymptote at $x = -\dfrac{5}{3}$.

There is a horizontal asymptote at $y = -\dfrac{2}{3}$.

The y-intercept, when $x = 0$, is $y = \dfrac{7}{5}$.

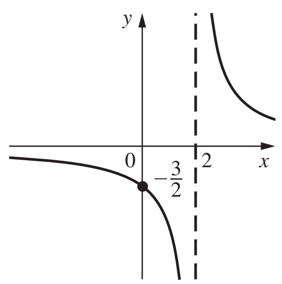


Example

(a) Sketch the curve $f(x) = \frac{3}{x - 2}$.

(b) Solve $\frac{3}{x - 2} < 4, x > 0$.

(a) For $f(x)$, we know that the graph has a vertical asymptote at $x = 2$ and will have a horizontal asymptote at $y = 0$.



(b) In order to solve this inequality, we first solve $f(x) = 4$

$$\Rightarrow \frac{3}{x - 2} = 4$$
$$\Rightarrow 4x - 8 = 3$$
$$\Rightarrow 4x = 11$$
$$\Rightarrow x = \frac{11}{4}$$

Using the graph, the solution to the inequality is $x > \frac{11}{4}$.

Exercise 7

1 Sketch (a) $y = 2\cos 3x°$ (b) $y = 4\sin\left(\theta + \frac{\pi}{3}\right)$ (c) $y = -3\sin\theta + 2$

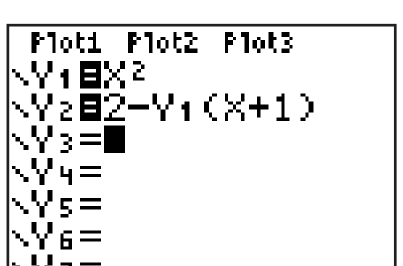
2 Sketch (a) $y = 3x^2$ (b) $y = (x - 2)^2$ (c) $y = 8 - x^2$

3 For each function $f(x)$, sketch (i) $y = f(x)$ (ii) $y = f(x - 2)$ (iii) $y = f(x) - 1$ (iv) $y = -2f(x)$

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = 3x$ **d** $f(x) = 4 - x$

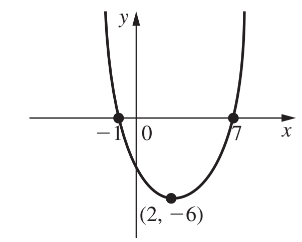
e $f(x) = \frac{1}{x}$ **f** $f(x) = x^2 - 3$ **g** $f(x) = \frac{3}{x - 4}$ **h** $f(x) = \frac{x + 2}{x - 1}$

4 For each of the functions in question 3, find an expression for $2 - f(x + 1)$ algebraically. Sketch each graph of $2 - f(x + 1)$ using a graphing calculator, thus checking your answer.

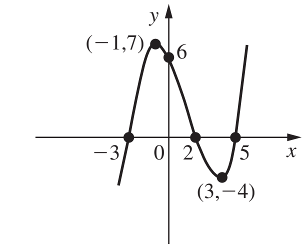


5 For each graph $y = f(x)$, sketch (i) $f(x + 3)$ (ii) $f(-x)$ (iii) $5 - 3f(x)$

a



b



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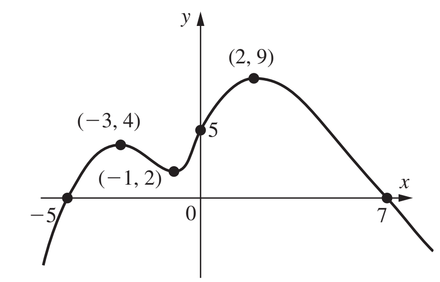

6 Sketch the graph of the rational function $y = f(x)$.

a $f(x) = \frac{2}{x}, x \neq 0$ **b** $f(x) = \frac{1}{x - 3}, x \neq 3$

c $f(x) = \frac{1}{x + 2}, x \neq -2$ **d** $f(x) = \frac{3}{x - 4}, x \neq 4$

e $f(x) = \frac{5}{x + 7}, x \neq -7$ **f** $f(x) = \frac{1}{2x + 1}, x \neq -\frac{1}{2}$

g $f(x) = \frac{6}{2x - 3}, x \neq \frac{3}{2}$ **h** $f(x) = \frac{1}{7 - x}, x \neq 7$

i $f(x) = \frac{-4}{x + 5}, x \neq -5$ **j** $f(x) = \frac{3}{8 - 5x}, x \neq \frac{8}{5}$

7 Sketch the graph of the rational function $y = g(x)$.

a $g(x) = \frac{x + 6}{x - 1}, x \neq 1$ **b** $g(x) = \frac{x - 4}{x + 3}, x \neq -3$

c $g(x) = \frac{2x + 1}{x - 6}, x \neq 6$ **d** $g(x) = \frac{8 - x}{x + 2}, x \neq -2$

e $g(x) = \frac{9 - x}{3 - x}, x \neq 3$ **f** $g(x) = \frac{5x - 2}{2x + 1}, x \neq -\frac{1}{2}$

g $g(x) = \frac{10x - 1}{2x + 3}, x \neq -\frac{3}{2}$ **h** $g(x) = \frac{7x + 2}{2x - 3}, x \neq \frac{3}{2}$

8 Solve the following equations for $x \in \mathbb{R}$.

a $\frac{10}{x + 2} = 4$ **b** $\frac{7}{x - 1} = 3$ **c** $\frac{3}{2x + 1} = 5$

d $\frac{8}{2 - x} = 6$ **e** $\frac{9}{7 - 2x} = -4$

9 Solve the following inequalities for $x \in \mathbb{R}, x > 0$.

a $\frac{1}{x + 5} < 6$ **b** $\frac{1}{x - 3} \leq 4$ **c** $\frac{2}{x + 3} < 7$

d $\frac{8}{2x - 1} < 1$ **e** $\frac{8}{3x - 2} < 4$ **f** $\frac{2}{3 - x} \geq -2$

g $\frac{x + 2}{x - 1} < 5$ **h** $\frac{6x + 1}{2x - 1} < 5$

10 Use your graphing calculator to draw a sketch of
$f(x) = x^2(x-2)^3$, $g(x) = f(-x) + 3$ and $h(x) = |g(x)|$.

11 Use your graphing calculator to draw a sketch of $f(x) = \frac{x^2+1}{x-3}$ for $x \neq 3$, $g(x) = f(x-2)$ and $h(x) = |g(x)|$.

12 Use your graphing calculator to draw a sketch of $p(x) = g(f(x))$ given $f(x) = x^2 - x - 6$ and $g(x) = 3x - 1$. Hence draw the graph of $p(|x|)$.

Review exercise

1 For $f(x) = \frac{3x^2-5}{x}$, find $f(2)$.

2 For $g(x) = \frac{2x+1}{x-2}$ with domain $\{-4, 0, 1, 5\}$, draw an arrow diagram and state the range.

3 For $f(x) = 7x - 4$, find

a $f(3x)$ **b** $f(2x-1)$ **c** $f\left(\frac{1}{x}\right)$

4 For $f(x) = 8 - 3x$ and $g(x) = \frac{x}{x-1}$, $x \neq 1$ find

a $f(g(x))$ **b** $g(f(x))$ **c** $f(f(x))$ **d** $g(g(x))$

5 For each function $f(x)$, choose a suitable domain so that an inverse exists and find $f^{-1}(x)$.

a $f(x) = x^2 - 6$ **b** $f(x) = \frac{1}{x+5}$ **c** $f(x) = \frac{7}{2x+3}$

6 Sketch the graph of $f(x) = \frac{2}{x+1}$ and its inverse function $f^{-1}(x)$.

7 For $f(x) = x^2 + 4x - 12$, sketch the graph of $y = |f(x)|$ and $y = f(|x|)$.

8 Solve $|2x + 9| = 7$.

9 Solve $|7 - 5x| < 3$.

10 Sketch the graph of $y = \frac{2}{x+3}$, indicating asymptotes and the y-intercept.

11 Sketch the graph of $y = \frac{x+2}{x^2+4x-9}$, indicating asymptotes, roots, y-intercept and turning points.

12 Sketch the graph of $y = \frac{\cos x}{3x^2}$, indicating asymptotes, roots, y-intercept and turning points.

13 For $f(x) = 2x + 1$, sketch the graph of

a $f(x-3)$ **b** $f(3x)$ **c** $4 - 5f(x)$

14 Sketch the graph of each of these rational functions:

a $f(x) = \frac{7}{2x+1}$ **b** $f(x) = \frac{-4}{3x+2}$ **c** $f(x) = \frac{8x-3}{2x+1}$

15 Solve $\frac{9}{3x-2} = 5$.

16 Solve $\frac{2}{4x-1} \leq 3$ for $x > 0$.

17 Use your graphing calculator to draw a sketch of
$f(x) = \frac{x^3-5x+1}{x+2}$, $g(x) = f(x+3)$ and $h(x) = |g(x)|$.

18 The one-to-one function f is defined on the domain $x > 0$ by $f(x) = \frac{2x-1}{x+2}$.

a State the range, A, of f. **b** Obtain an expression for $f^{-1}(x)$, for $x \in A$.
[IB May 02 P1 Q15]

19 Solve the inequality $|x - 2| \geq |2x + 1|$. [IB May 03 P1 Q13]

20 A function f is defined for $x \leq 0$ by $f(x) = \frac{x^2-1}{x^2+1}$.

Find an expression for $f^{-1}(x)$. [IB May 03 P1 Q17]

21 Let $f: x \to \sqrt{\frac{1}{x^2} - 2}$.

Find

a the set of real values of x for which f is real and finite **b** the range of f.
[IB May 01 P1 Q5]

22 Let $f(x) = \frac{x+4}{x+1}$, $x \neq -1$ and $g(x) = \frac{x-2}{x-4}$, $x \neq 4$.

Find the set of values of x such that $f(x) \leq g(x)$. [IB Nov 03 P1 Q17]