10 Differentiation 3 – Applications

Differential calculus is widely used in both the natural sciences and the human sciences. In physics, if we want to investigate the speed of a body falling under gravity, the force that will give a body a certain acceleration and hence a certain velocity, or the rate of decay of a radioactive material, then differential calculus will help us. In chemistry, we determine rates of reaction using calculus, and in biology a problem such as the rate of absorption of aspirin into the bloodstream as a function of time would have its solution based on differential calculus.

Differentiation may also be applied to a large number of problems that deal with the issue of extremes; this could include the biggest, the smallest, the greatest or the least. These maximum or minimum amounts may be described as values for which a certain rate of change (increase or decrease) is zero, i.e. stationary points. For example, it is possible to determine how high a projectile will go by finding the point at which its change of altitude with respect to time, that is, its velocity, is equal to zero.

10.1 Optimization problems

We have already met the idea of stationary points on a curve, which can give rise to local maxima and minima. This idea of something having a maximum and a minimum value can be used in a variety of situations. If we need to find out when a quantity is as small as possible or as large as possible, given that we can model the situation mathematically, then we can use differential calculus.

Imagine a car manufacturing company that is aiming its new model at the cheaper end of the market. One of the jobs of the marketing department in this company is to decide how much to sell each car for. If they decide to sell at just above cost price, then the company will only make a small amount of profit per car, but provided all other features of the marketing are correct the company will sell a large number. If they decide to charge a higher price, then the company will make more profit per car, but will probably sell fewer cars. Hence the marketing department need to find the right price to charge that will maximize the company's profit.

Obviously to maximize a function f(x) you are looking for the greatest value within a given region. Similarly to minimize a function, you are looking for the smallest. This may or may not be a stationary point. Many economists and engineers are faced with problems such as these, and this area of study is known as **optimization**. For the purposes of this course the greatest and least values will always occur at a stationary point.

Because it is based on "real life" situations, the problem is not always as simple as it might first seem. At its most basic level, optimization is just applying differentiation to a formula given in a "real life" scenario. However, in more complex questions there are often a number of steps that may need to precede this.

Method for solving optimization problems

1 Draw a diagram and write down the formula suggested by the question.
2 If the formula involves three variables, find a link between two of them.
3 Now substitute into the original formula. We now have a formula that links two variables.
4 Differentiate.
5 Make the equation equal to zero and solve.
6 Find the other value(s) by substitution.
7 Check whether it is a maximum or minimum point.

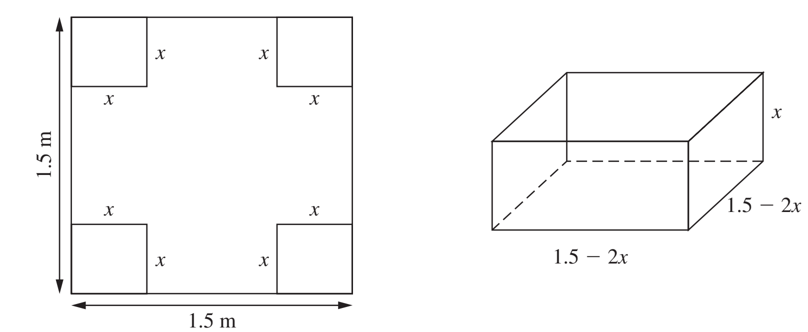
Always check that it is the two variables that the question is talking about!

We will now demonstrate this with a number of examples.

Example

In design technology class, Ayesha is asked to make a box in the shape of a cuboid from a square sheet of card with each edge being 1.5 metres long. To do this she removes a square from each corner of the card. What is the biggest box that she can make?

The diagram below shows the cardboard with a square of side x metres removed from each corner, and the box into which it is made.



Step 1. Volume of the box, $V = x(1.5 - 2x)^2$. In this case the formula has only two variables: hence steps 2 and 3 can be ignored.

Step 4.

$$V = x(2.25 - 6x + 4x^2)$$
$$= 2.25x - 6x^2 + 4x^3$$
$$\Rightarrow \frac{dV}{dx} = 2.25 - 12x + 12x^2$$

Step 5. This is stationary when $2.25 - 12x + 12x^2 = 0$.

Using a calculator to solve the quadratic equation gives $x = \frac{1}{4}$ or $x = \frac{3}{4}$.

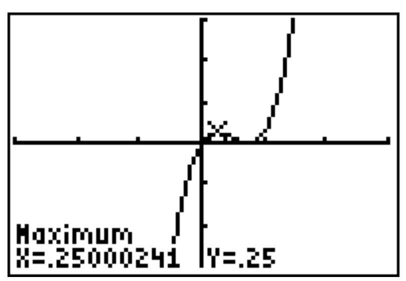
Step 6. $V = \frac{1}{4}$ or 0

For step 7, to test that this is indeed a maximum, we differentiate $\frac{dV}{dx}$ and apply the second derivative test.

$$\frac{d^2V}{dx^2} = -12 + 24x$$

When $x = \frac{1}{4}, \frac{d^2V}{dx^2} = -6$ and when $x = \frac{3}{4}, \frac{d^2V}{dx^2} = 6$. Hence the maximum value occurs when $x = \frac{1}{4}$ and the maximum volume is $\frac{1}{4}\text{m}^3$.

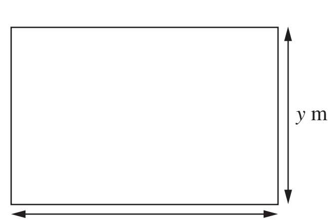
This question can also be done on a calculator by inputting the curve $y = x(1.5 - 2x)^2$ and finding the maximum value.
The calculator display is shown below.



Hence the maximum volume is $\frac{1}{4}\text{m}^3$.

Example

A farmer wishes to fence in part of his field as a safe area for his sheep. The shape of the area is a rectangle, but he has only 100 m of fencing. What are the dimensions of the safe area that will make it as large as possible?
The area is a rectangle with dimension *x* m by *y* m.
This is shown below.



Step 1. Area $= xy$, i.e. $A = xy$. Hence we have a formula with three variables.
Step 2. We know that the farmer has 100 m of fencing, hence $2x + 2y = 100$.

Therefore $y = \frac{100 - 2x}{2} = 50 - x$

Step 3. Find a formula for the area in terms of *x*. In this case it does not matter if we substitute for *x* or for *y*, as we need to find both in the end.
So $A = x(50 - x)$
$\Rightarrow A = 50x - x^2$

Step 4. $\frac{dA}{dx} = 50 - 2x$

Step 5. This is stationary when $50 - 2x = 0$
$\Rightarrow x = 25$

Step 6. $y = 25$ and therefore $A = 25^2 = 625$

For step 7, to test that this is indeed a maximum, we differentiate $\frac{dA}{dx}$ and apply the second derivative test.

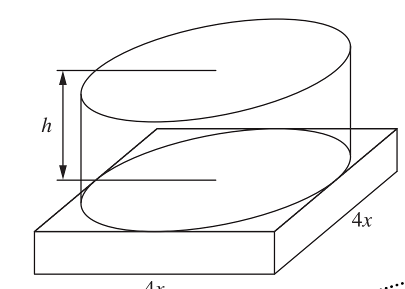
$$\frac{d^2A}{dx^2} = -2$$

Since it is negative, the area of 625 m^2 is a maximum value. Hence the maximum value is given when $x = y = 25$ m. This should come as no surprise, as the maximum area of any rectangle is when it is a square.

Even though it is obvious that the value given is a maximum, it is still important that we demonstrate it.

Example

The diagram shows a solid body made from a cylinder fixed to a cuboid. The cuboid has a square base with each edge measuring $4x$ cm and a height of x cm. The cylinder has a height of h cm, and the base of the cylinder fits exactly on the cuboid with no overlap. Given that the total volume of the solid is 80 cm^3, find the minimum surface area.



Step 1.
(Surface area) $A = \overbrace{4x^2 + 4x^2 + 4x^2 + 4x^2} + \overbrace{16x^2 + 16x^2}$
$+ \underbrace{\pi(2x)^2 - \pi(2x)^2} + 2\pi\underbrace{(2x)h}$

These are the areas of the four sides.

This is the area of the top and bottom.

These are the areas of the circles. You need to add on the top one, but subtract the bottom, as the area showing is a square minus a circle. The radius of the circle is $2x$.

This is the curved surface area of the cylinder.

Hence we have a formula with three variables:
$A = 48x^2 + 4\pi xh$

Step 2. We know that the volume of the body is 80 cm^3, hence:
$16x^3 + \pi(2x)^2h = 80$

$$\Rightarrow h = \frac{80 - 16x^3}{4\pi x^2}$$

Step 3.

$$A = 48x^2 + 4\pi x\left(\frac{80 - 16x^3}{4\pi x^2}\right)$$

$$A = 48x^2 + \frac{80}{x} - 16x^2$$

$$\Rightarrow A = 32x^2 + \frac{80}{x}$$

Step 4.

$$\frac{dA}{dx} = 64x - \frac{80}{x^2}$$

Step 5. This is stationary when

$$64x - \frac{80}{x^2} = 0$$

$$\Rightarrow 64x^3 = 80$$

$$\Rightarrow x = \sqrt[3]{\frac{80}{64}} = 1.07\ldots$$

Step 6 allows us to find that $A = 111\ldots$ and $h = 4.11\ldots$

For step 7, to test that this is indeed a maximum, differentiate $\frac{dA}{dx}$ and apply the second derivative test.

$$\frac{dA}{dx} = 64x - 80x^{-2}$$

$$\frac{d^2A}{dx^2} = 64 + \frac{160}{x^3}$$

So when $x = 1.07\ldots$, $\frac{d^2A}{dx^2} = 192\ldots$. Since it is positive the area of 111 cm^2 is a minimum value.

Example

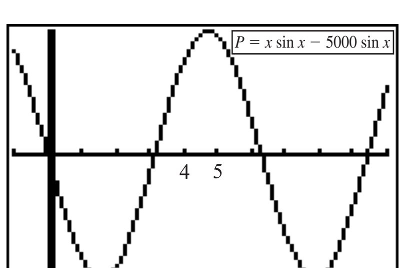
The marketing director for MacKenzie Motors has calculated that the profit made on each car is given by the formula $P = xy - 5000y$ where x is the selling price of each car (in thousands of pounds sterling) and y is the number of cars sold, which is affected by the time of year. Given that x and y are related by the formula $y = \sin x$, find the maximum profit that can be made.

Step 1. $P = xy - 5000y$.

Step 2. The link between two of the variables is $y = \sin x$.

Step 3. $P = x \sin x - 5000 \sin x$.

Steps 4, 5 and 6. This can be differentiated, but the resulting equation will need to be solved on a calculator. Hence it will be more effective to find the maximum value of the curve at this stage. Because this is a sinusoidal curve, we need to find where the first maximum occurs.



Hence we know the maximum value of $P = 4995.27\ldots$ and occurs when $x = 4.71$.

For step 7 it is not sensible to do a second derivative test. Instead we put in a sketch of the curve above and state that the y-values either side of the maximum are less than the maximum. It is important that actual values are given.

When $x = 4.70$, $P = 4994.91\ldots$

and when $x = 4.72$, $P = 4995.14\ldots$ which are both less than $4995.27\ldots$ and hence the maximum value of $P = £4995$.

Exercise 1

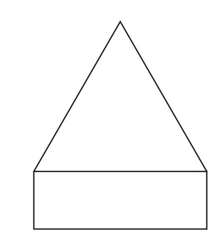
1 The amount of power a car engine produces is related to the speed at which the car is travelling. The actual relationship is given by the formula $P = 15v + \frac{7500}{v}$. Find the speed when the car is working most efficiently, i.e. when the power is the least.

2 A cuboid has a square base and a total surface area of 300 cm^2. Find the dimensions of the cuboid for the volume to be a maximum.

3 The base for a table lamp is in the shape of a cylinder with one end open and one end closed. If the volume of the base needs to be 1000 cm^3, find the radius of the base such that the amount of material used is a minimum.

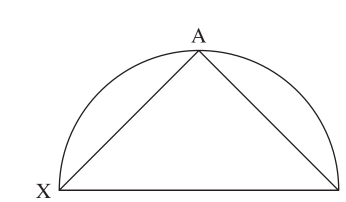
4 For stacking purposes, a manufacturer of jewellery boxes needs to make them in the shape of a cuboid where the length of the box must be three times the width. The box must have a capacity of 400 cm^3. Find the dimensions of the box that would have the smallest surface area.

5 The diagram below shows a rectangle with an equilateral triangle on top. If the perimeter of the shape is 28 cm, find the length of the sides of the rectangle such that the area of the shape is a maximum.

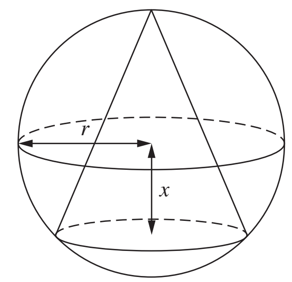


6 One of the clients of a packaging company is a soup manufacturer who needs tin cans manufactured. To maximize the profit, the surface area of the can should be as small as possible. Given that the can must hold 0.25 litres and is cylindrical, find the minimum surface area.

7 Consider the semicircle below. It has diameter XY and the point A is any point on the arc XY. The point A can move but it is required that $XA + AY = 25$. Find the maximum area of the triangle XAY.



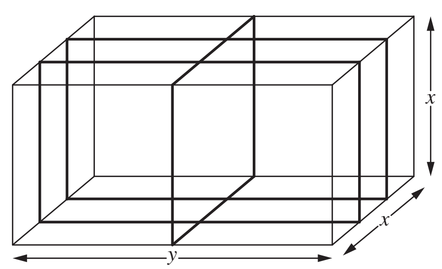
8 A cone is cut from a sphere as shown below. The radius of the sphere is r and x is the distance of the base of the cone from the centre of the sphere.



a Prove that the volume V of the cone is $V = \frac{\pi}{3}(r - x)(r + x)^2$.

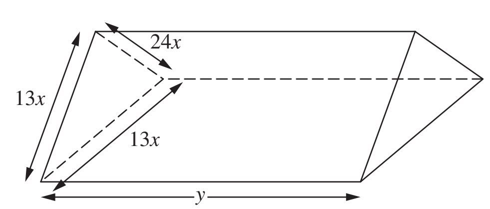
b Find the height of the cone when the volume of the cone is a maximum.

9 A courier company requires that parcels be secured by three pieces of string. David wants to send a parcel in the shape of a cuboid. The cuboid has square ends. The square is of side x cm and the length of the parcel is y cm. This is shown in the diagram below.



Given that the total length of string used is 900 cm, find the volume of the parcel in terms of x. Hence find the values of x and y for which the volume has a stationary value and determine whether this is a maximum or a minimum.

10 An open container is made from four pieces of sheet metal. The two end pieces are both isosceles triangles with sides of length $13x$, $13x$ and $24x$ as shown below. The other two pieces that make up the container are rectangles of length y and width $13x$. The total amount of sheet metal used is 900 cm^2.



a Show that $y = \frac{450 - 60x^2}{13x}$.

b Find the volume of the container in terms of x.

c Find the value of x for which the volume of the container, V, has a stationary value and determine whether this is a maximum or a minimum.

11 In an intensive care unit in hospital, the drug adrenalin is used to stabilize blood pressure. The amount of the drug in the bloodstream y at any time t is the combination of two functions, $P(t)$ and $Q(t)$, which takes into consideration the fact that the drug is administered into the body repeatedly. Researchers at the hospital have found that $P(t) = e^{-t}\sin t$, while the manufacturers of the drug have found that $Q(t) = \cos t + 2$. The researchers have also found that $y = P(t) + Q(t)$. Given that the drug is initially administered at time $t = 0$, find the first two times when the quantity of drug in the bloodstream is the greatest, and verify using differentiation that these are in fact maximum values.

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12 The population P, in thousands, of mosquitoes in the Kilimanjaro region of Tanzania over a 30-day period in May is affected by two variables: the average daily rainfall, r, and the average daily temperature, θ. The rainfall is given by the function $r = t\cos t + 6$ and the temperature is given by $\theta = e^{\frac{t}{10}} + 7$, where t is the time in days. It has been found that $P = r + \theta$. Find the minimum number of mosquitoes after the first five days of May and verify that it is a minimum.

13 In Japan the running cost of a car in yen per hour, Y, is dependent on its average speed v. This is given by the formula $Y = 6 + \frac{v^2 + 1}{v - 1}$, where v is the speed in tens of kilometres per hour. Write down the cost of a journey of 200 km covered at an average speed of 50 kmh^{-1} and find the speed that would make the cost of this journey a minimum.

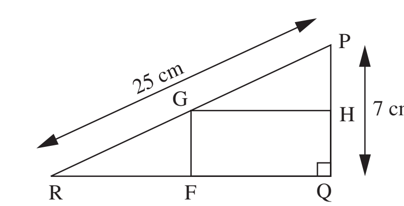
14 A new car hire company, Bob's Rentals, has just opened and wants to make the maximum amount of profit. The amount of profit, P, is dependent on two factors: x, the number of tens of cars rented; and y, the distance travelled by each car. The profit is given by the formula $P = xy + 40$, where $y = \sin x$. Find the smallest number of cars that the company needs to rent to give the maximum amount of profit.

15 In a triangle PQR, angle PQR is 90°, PQ = 7 cm and PR = 25 cm. The rectangle QFGH is such that its vertices F, G and H lie on QR, PR and PQ respectively. This is shown below.

a Given that QH = x cm and GH = y cm, find a relationship between x and y.

b Hence express the area of the rectangle in terms of x only.

c Calculate the maximum value of this area as x varies.



10.2 Rates of change of connected variables

Consider a variable x. If the rate of change of the variable x is 2 ms^{-1}, then what we mean is that the rate of change of x with respect to time is 2, i.e. $\frac{dx}{dt} = 2$. The units of the variable give this information. Now, sometimes we want to find the rate of change with respect to time of a variable that is connected to x, say y, where $y = x^2$, at the point when $x = 10$. To do this, we use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. In this case we would use the formula $\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$.

Since $\frac{dy}{dx} = 2x$

$\Rightarrow \frac{dy}{dx} = 20.$

Therefore $\frac{dy}{dt} = 2 \times 20 = 40\ \text{ms}^{-1}$.

This is known as a **connected rate of change**.

Method for finding connected rates of change

This occurs when a question asks for a rate of change of one quantity but does not give a direct equation, and hence it is necessary to make a connection to another equation.

1. Write down the rate of change required by the question.
2. Write down the rate of change given by the question.
3. Write down an expression that connects the rate of change required and the one given.
4. This connection produces a third rate of change, which needs to be calculated. Find an equation that will give this new rate of change.
5. Differentiate the new equation.
6. Multiply the two formulae together and substitute to find the required rate of change.

Example

The radius, r, of an ink spot is increasing at the rate of $2\ \text{mms}^{-1}$. Find the rate at which the area, A, is increasing when the radius is 8 mm.

Step 1. The rate of change required is $\frac{dA}{dt}$.

Step 2. The known rate of change is $\frac{dr}{dt} = 2$.

Step 3. The connection is $\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$.

Step 4. We now need an equation linking A and r. For a circle $A = \pi r^2$.

Step 5. $\frac{dA}{dr} = 2\pi r$

Step 6. $\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$

$\Rightarrow \frac{dA}{dt} = 2 \times 2\pi r$

$\Rightarrow \frac{dA}{dt} = 4\pi r$

$\Rightarrow \frac{dA}{dt} = 4 \times \pi \times 8$

$\Rightarrow \frac{dA}{dt} = 32\pi$

Before we proceed with further examples we need to establish the result that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

We know that $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

Since $\frac{\delta y}{\delta x}$ is a fraction $\frac{dx}{dy} = \lim_{\delta x \to 0} \frac{1}{\frac{\delta y}{\delta x}}$.

However, as $\delta y \to 0$, $\delta x \to 0$.

Therefore $\frac{dx}{dy} = \lim_{\delta y \to 0} \frac{1}{\frac{\delta y}{\delta x}}$, giving the result that

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Remember δy and δx are numbers whereas $\frac{dy}{dx}$ is a notation.

Example

A spherical balloon is blown up so that its volume, V, increases at a constant rate of $3\ \text{cm}^3\text{s}^{-1}$. Find the equation for the rate of increase of the radius r.

Step 1. The rate of change required is $\frac{dr}{dt}$.

Step 2. The known rate of change is $\frac{dV}{dt} = 3$.

Step 3. The connection is $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$.

Step 4. We now need an equation linking V and r. For a sphere $V = \frac{4}{3}\pi r^3$. It is much easier to find $\frac{dV}{dr}$ than $\frac{dr}{dV}$, and hence we use the rule above.

Step 5. $\frac{dV}{dr} = 4\pi r^2$.

Step 6. $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$\frac{dr}{dt} = 3 \times \frac{1}{4\pi r^2} = \frac{3}{4\pi r^2}$

Example

The surface area, A, of a cube is increasing at a rate of $20\ \text{cm}^2\,\text{s}^{-1}$. Find the rate of increase of the volume, V, of the cube when the edge of the cube is 10 cm.

Step 1. The rate of change required is $\frac{dV}{dt}$.

This is rather different from what has been asked previously, where questions have required that we differentiate the dependent variable directly with respect to what is known as the independent variable. In the case of $\frac{dy}{dx}$, y is the dependent variable and x is the independent variable. However in a case like this the independent variable is t since all the other variables are dependent on this. Which variable is the independent one is not always immediately obvious.

Step 2. The known rate of change is $\frac{dA}{dt} = 3$.

Step 3. The connection is $\frac{dV}{dt} = \frac{dA}{dt} \times \frac{dV}{dA}$.

A formula linking volume and area is not very straightforward, and nor is differentiating it. Hence we now connect the volume and area using the length of an edge, x.

This gives $\frac{dV}{dA} = \frac{dV}{dx} \times \frac{dx}{dA}$

which leads to the formula $\frac{dV}{dt} = \frac{dA}{dt} \times \frac{dV}{dx} \times \frac{dx}{dA}$.

Step 4. We now need equations linking V and x and A and x. For a cube $V = x^3$ and $A = 6x^2$.

Step 5. $\frac{dV}{dx} = 3x^2$ and $\frac{dA}{dx} = 12x$

Step 6. $\frac{dV}{dt} = \frac{dA}{dt} \times \frac{dV}{dx} \times \frac{dx}{dA} = 20 \times 3x^2 \times \frac{1}{12x} = 5x$

So when $x = 10$, $\frac{dV}{dt} = 50\ \text{cm}^3\,\text{s}^{-1}$.

Now steps 5 and 6 can be done in an alternative way.

Step 5. $\frac{dV}{dx} = 3x^2$ and $\frac{dA}{dx} = 12x$

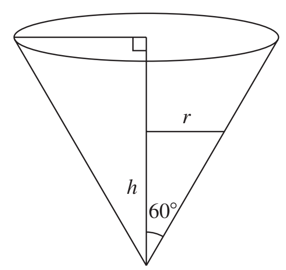
When $x = 10$, $\frac{dV}{dx} = 300$ and $\frac{dA}{dx} = 120$.

Step 6. $\frac{dV}{dt} = \frac{dA}{dt} \times \frac{dV}{dx} \times \frac{dx}{dA} = 20 \times 300 \times \frac{1}{120} = 50\ \text{cm}^3\,\text{s}^{-1}$

The method to use is personal choice, although if a formula is required, then the first method must be used.

Example

Water is being poured into a cone, with its vertex pointing downwards. This is shown below. The cone is initially empty and water is poured in at a rate of $25\ \text{cm}^3\,\text{s}^{-1}$. Find the rate at which the depth of the liquid is increasing after 30 seconds.



Step 1. The rate of change required is $\frac{dh}{dt}$, where h is the depth of the liquid.

Step 2. The known rate of change is $\frac{dV}{dt} = 25$.

Step 3. The connection is $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$.

Step 4. We need to find a formula linking V and h. For a cone $V = \frac{1}{3}\pi r^2 h$. We now need to find a formula that connects h and r. From the diagram

$\tan 60° = \frac{r}{h}$

$\Rightarrow r = h\sqrt{3}$

$\Rightarrow V = \frac{1}{3}\pi \times 3h^2 \times h$

$\Rightarrow V = \pi h^3$

Step 5. $\frac{dV}{dh} = 3\pi h^2$.

Step 6. $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$\Rightarrow \frac{dh}{dt} = 25 \times \frac{1}{3\pi h^2} = \frac{25}{3\pi h^2}$

After 30 seconds, the volume will be $30 \times 25 = 750\ \text{cm}^3$. Using $V = \pi h^3$ gives $750 = \pi h^3$

$\Rightarrow h = 6.20\ldots$

$\Rightarrow \frac{dh}{dt} = \frac{25}{3\pi \times 6.20^2} = 0.0689\ \text{cm}\,\text{s}^{-1}$

Example

A point P moves in such a way that its coordinates at any time t are given by $x = te^{2\sin t}$ and $y = e^{-2\cos t}$. Find the gradient of the line OP after 5 seconds.

Step 1. The rate of change required is $\frac{dy}{dx}$.

Step 2 is not required.

Step 3. The connection is $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$.

Step 4 is not required.

Step 5. $x = te^{2\sin t}$

$\Rightarrow \frac{dx}{dt} = e^{2\sin t} + 2t\cos te^{2\sin t}$

When $t = 5$,

$\frac{dx}{dt} = e^{2\sin 5} + 2 \times 5\cos 5e^{2\sin 5}$

$= 0.563\ldots$

and

$y = e^{-2\cos t}$

$\Rightarrow \frac{dy}{dt} = 2\sin te^{-2\cos t}$

When $t = 5$,

$\frac{dy}{dt} = 2\sin 5e^{-2\cos 5}$

$= -1.08\ldots$

Step 6.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = -1.09 \times \frac{1}{0.564}$

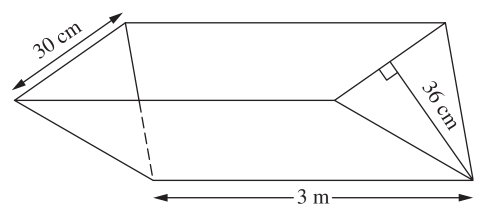
$\Rightarrow \frac{dy}{dx} = -1.93$

Exercise 2

1 The surface area of a sphere is given by the formula $A = 4\pi r^2$, where r is the radius. Find the value of $\frac{dA}{dr}$ when $r = 3$ cm. The rate of increase of the radius is $4\,\mathrm{cm\,s^{-1}}$. Find the rate of increase of the area when $r = 4$ cm.

2 Orange juice is being poured into an open beaker, which can be considered to be a cylinder, at a rate of $30\,\mathrm{cm^3\,s^{-1}}$. The radius of the cylinder is 8 cm. Find the rate at which the depth of the orange juice is increasing.

3 The cross-sectional area of a trough is an isosceles triangle of height 36 cm and base 30 cm. The trough is 3 m long. This is shown below.



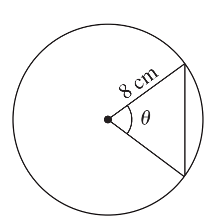
If water flows into the trough at a rate of $500\,\mathrm{cm^3\,s^{-1}}$, find the rate at which the water level is increasing when the height is h.

4 The population, P, of termites varies with time t hours according to the formula $P = N_0 e^{3m}$ where N_0 is the initial population of termites and m is a variable given by $m = 3e^{\sin t}$. Find the rate of change of the termite population after 6 hours, giving your answer in terms of N_0.

5 A wine glass has been made such that when the depth of wine is x, the volume of wine, V, is given by the formula $V = 3x^3 - \frac{1}{3x}$. Alexander pours wine into the glass at a steady rate, and at the point when its depth is 4 cm, the level is rising at a rate of $1.5\,\mathrm{cm\,s^{-1}}$. Find the rate at which the wine is being poured into the glass.

6 Bill, who is 1.85 m tall, walks directly away from a street lamp of height 6 m on a level street at a velocity of $2.5\,\mathrm{m\,s^{-1}}$. Find the rate at which the length of his shadow is increasing when he is 4 m away from the foot of the lamp.

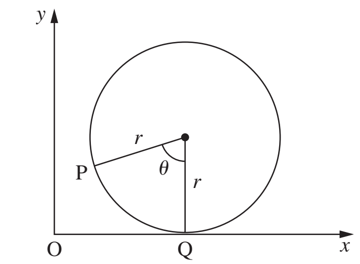
7 Consider the segment of a circle of fixed radius 8 cm. If the angle θ increases at a rate of 0.05 radians per second, find the rate of increase of the area of the segment when $\theta = 1.5$ radians.



8 An empty hollow cone of radius a and height $4a$ is held vertex downwards and water is poured in at a rate of $8\pi\,\mathrm{cm^3\,s^{-1}}$. Find the rate at which the depth of water is increasing after 25 seconds.

9 A point P moves in such a way that its coordinates at any time t are given by $x = \frac{1}{1+t^2}$ and $y = \tan^{-1} t$. Find the gradient of the line OP after 3 seconds.

10 A circular disc of radius r rolls, without slipping, along the x-axis. The plane of the disc remains in the plane Oxy. A point P is fixed on the circumference of the disc and is initially at O. When the disc is rolled through θ radians, the point of contact is now Q and the length of the arc PQ is now the same as OQ. This is shown below.



a Find the coordinates of P in terms of r and θ.

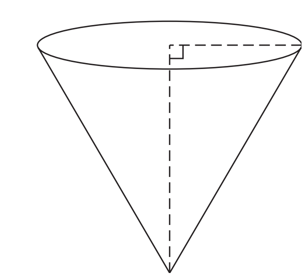
b Using connected rates of change, show that the gradient of the curve is $\cot\frac{\theta}{2}$.

11 A balloon is blown up so that its surface area is increasing at a rate of $25\,\mathrm{cm^2\,s^{-1}}$. What is the rate of increase of the volume when its radius is 8 cm? Assume the balloon is spherical at all times.

12 A point moves on a curve such that $x = e^{3t}\cos 3t$ and $y = e^{3t}\sin 3t$, where t is the time taken. Show that the gradient at any time t is given by the formula

$$\frac{dy}{dx} = \tan\left(3t + \frac{\pi}{4}\right).$$

13 Water is being poured into a cone, with its vertex pointing downwards. This is shown below.



The cone is initially empty and water is poured in at a rate of $2\pi\sqrt{3}\,\mathrm{cm^3\,s^{-1}}$. Find the rate of increase of:

a the radius of the circular surface of the water after 4.5 seconds

b the area of the circular surface of the water after 4.5 seconds.

10.3 Displacement, velocity and acceleration

This is another important application of differential calculus, which goes back to the basic definitions met in Chapter 8. Usually we define s as the displacement, v as the velocity, and a as the acceleration. If we consider a body moving 100 m in 25 seconds, some very basic knowledge will tell us that its average speed is $\frac{100}{25}\,\mathrm{m\,s^{-1}}$, i.e. total distance travelled divided by total time taken. However, unless the body keeps a constant speed we have no idea what the velocity was after 4 seconds. In order to deal with this we now deal with velocity in a different way. By definition, velocity is the rate of change of displacement with respect to time. In questions like this displacement and distance are often used interchangeably, but because direction matters, we technically need to use a vector quantity and hence we usually talk about displacement. The distinction between vector and scalar quantities is discussed in Chapter 12. Since the differential operator $\frac{d}{dt}$ means "rate of change with respect to time", we now find that velocity, $v = \frac{ds}{dt}$. Using a similar argument, acceleration $a = \frac{dv}{dt}$ or alternatively $\frac{d^2s}{dt^2}$. Hence if we have a displacement formula as a function of time, we can now work out its velocity and acceleration.However, what happens when acceleration is related to displacement? We know that $a = \frac{dv}{dt}$. From the work on connected rates of change, $\frac{dv}{dt}$ could be written as $\frac{ds}{dt} \times \frac{dv}{ds}$. However, $\frac{ds}{dt}$ is actually v. Hence $a = v\frac{dv}{ds}$. This now gives a formula that links velocity and displacement.

To summarize:

Quantity	Notation
Velocity	$\frac{ds}{dt}$
Acceleration	$\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$ or $v\frac{dv}{ds}$

Example

If the displacement of a particle is given by the formula $s = 3t^3 - 20t^2 + 40t$, find:

a) the displacement after 3 seconds
b) the formula for the velocity at any time t
c) the values of t when the particle is not moving
d) the initial velocity of the particle
e) the formula for the acceleration at any time t
f) the initial acceleration of the particle.

a) When $t = 3$, $s = 3 \times 27 - 20 \times 9 + 40 \times 3 = 21$ m.
b) To find the velocity, differentiate the formula for s.

$$v = \frac{ds}{dt} = 9t^2 - 40t + 40$$

Hence $v = 9t^2 - 40t + 40$.
c) When the particle is not moving $v = 0$

$$\Rightarrow 9t^2 - 40t + 40 = 0$$
$$\Rightarrow (3t - 8)(3t - 5) = 0$$
$$\Rightarrow t = \frac{8}{3} \text{ secs or } t = \frac{5}{3} \text{ secs}$$

d) The initial velocity occurs when $t = 0$

$$\Rightarrow v = 40 \text{ ms}^{-1}$$

e) To find the acceleration, differentiate the formula for v

$$a = \frac{dv}{dt} = 18t - 40$$

f) The initial acceleration occurs when $t = 0$.

$$\Rightarrow a = -40 \text{ ms}^{-2}$$

The negative sign means it is a deceleration rather than an acceleration.

Example

A boat travels with variable speed. Its displacement at any time t is given by $s = 2t^3 - 8t^2 + 8t$. After how long in the journey:
a) is its displacement a maximum and what is its displacement at that point?
b) is its velocity a minimum and what is its velocity at that point?

a) To find the maximum displacement we differentiate:

$$s = 2t^3 - 8t^2 + 8t$$
$$v = \frac{ds}{dt} = 6t^2 - 16t + 8$$

For a maximum displacement

$$v = 6t^2 - 16t + 8 = 0$$

The maximum displacement occurs when the velocity is zero

$$\Rightarrow 3t^2 - 8t + 4 = 0$$
$$\Rightarrow (3t - 2)(t - 2) = 0$$
$$\Rightarrow t = \frac{2}{3} \text{ or } t = 2$$

Now $\frac{d^2s}{dt^2} = 6t - 8$.

Hence when $t = 2$, $\frac{d^2s}{dt^2} = 4$, which is positive and therefore a minimum.

When $t = \frac{2}{3}$, $\frac{d^2s}{dt^2} = -4$, which is negative and therefore a maximum.

When $t = \frac{2}{3}$, $s = \frac{64}{27}$ m.

b) To find the minimum velocity we use $\frac{dv}{dt}$, i.e. $\frac{d^2s}{dt^2}$.

$$v = 6t^2 - 16t + 8$$
$$\frac{dv}{dt} = 12t - 16$$

For a minimum velocity $12t - 16 = 0$

$$\Rightarrow t = \frac{4}{3} \text{ sec}$$

When $t = \frac{4}{3}$, $v = 6\left(\frac{4}{3}\right)^2 - 16\left(\frac{4}{3}\right) + 8 = -\frac{8}{3} \text{ ms}^{-1}$.

To test whether it is a minimum we find $\frac{d^2v}{dt^2}$.

Since $\frac{d^2v}{dt^2} = 12$, it is a minimum.

Example

The displacement of a particle is given by the equation $s = 5\cos\frac{\pi}{3}t + 10\sin\frac{\pi}{3}t$.

a) Give a formula for its velocity at any time t.
b) What is its initial velocity?
c) What is the minimum displacement of the particle?
d) Give a formula for its acceleration at any time t.

a) $s = 5\cos\frac{\pi}{3}t + 10\sin\frac{\pi}{3}t$

$$\frac{ds}{dt} = v = \frac{-5\pi}{3}\sin\frac{\pi}{3}t + \frac{10\pi}{3}\cos\frac{\pi}{3}t$$

b) The initial velocity is when $t = 0$.

$$v = \frac{-5\pi}{3}\sin 0 + \frac{10\pi}{3}\cos 0$$

$$\Rightarrow v = \frac{10\pi}{3}\text{ ms}^{-1}$$

c) The minimum displacement occurs when $\frac{ds}{dt} = 0$, i.e. $v = 0$.

$$\frac{-5\pi}{3}\sin\frac{\pi}{3}t + \frac{10\pi}{3}\cos\frac{\pi}{3}t = 0$$

$$\Rightarrow \frac{\sin\frac{\pi}{3}t}{\cos\frac{\pi}{3}t} = \frac{\frac{10\pi}{3}}{\frac{5\pi}{3}}$$

$$\Rightarrow \tan\frac{\pi}{3}t = 2$$

This equation has multiple solutions, but we only need to consider the first two positive solutions as after this it will just give repeated maxima and minima.

$$\Rightarrow \frac{\pi}{3}t = 1.10\ldots,\ 4.24\ldots$$

$$\Rightarrow t = 1.05\ldots,\ 4.05\ldots$$

$$\frac{d^2s}{dt^2} = \frac{-5\pi^2}{9}\cos\frac{\pi}{3}t - \frac{10\pi^2}{9}\sin\frac{\pi}{3}t$$

Now when, $t = 1.05\ldots$, $\frac{d^2s}{dt^2}$ is negative and when $t = 4.05\ldots$, $\frac{d^2s}{dt^2}$ is positive.

Hence $t = 4.05\ldots$ gives the minimum displacement. In this case $s = -11.2$ m.

d) The acceleration is given by $\frac{dv}{dt}$, i.e. $\frac{d^2s}{dt^2}$.

Hence $a = \frac{-5\pi^2}{9}\cos\frac{\pi}{3}t - \frac{10\pi^2}{9}\sin\frac{\pi}{3}t$.

Example

The displacement of a particle is given by the formula $s = \frac{\ln t}{t^2}$.

Find:
a) the formula for the velocity of the particle at any time t
b) the velocity of the particle after 3 seconds
c) the formula for the acceleration of the particle
d) the acceleration after 2 seconds.

a) $v = \frac{ds}{dt} = \frac{t^2\frac{1}{t} - 2t\ln t}{t^4} = \frac{1 - 2\ln t}{t^3}$

b) When $t = 3$, $v = \frac{1 - 2\ln 3}{27} = -0.443\text{ ms}^{-1}$.

The negative sign means the velocity is in the opposite direction.

c) $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{t^3\left(\frac{-2}{t}\right) - 3t^2(1 - 2\ln t)}{t^6}$

$$\Rightarrow a = \frac{t^2(-2 - 3 + 6\ln t)}{t^6}$$

$$\Rightarrow a = \frac{6\ln t - 5}{t^4}$$

d) When $t = 2$, $a = \frac{6\ln 2 - 5}{16} = -0.0526\text{ ms}^{-2}$.

Example

If the velocity of a particle is proportional to the square of the displacement travelled, prove that the acceleration is directly proportional to the cube of the displacement.

$v \propto s^2$
$\Rightarrow v = ks^2$

We know that the acceleration, a, is given by $v\frac{dv}{ds}$.

Now $\frac{dv}{ds} = 2ks$.

Therefore
$$a = ks^2 \times 2ks$$
$$= 2k^2s^3$$

Therefore the acceleration is directly proportional to the cube of the displacement as $2k^2$ is a constant.

Exercise 3

1 The displacement, s, travelled in metres by a bicycle moving in a straight line is dependent on the time, t, and is connected by the formula $s = 4t - t^3$.

a Find the velocity and the acceleration of the cyclist when $t = \frac{1}{2}$ sec.

b At what time does the cyclist stop?

2 If $v = 16t - 6t^2$ and the body is initially at O, find:

a the velocity when $t = 2$ secs

b an expression for the acceleration at any time t

c the acceleration when $t = 3$ secs.

3 The velocity of a car is dependent on time and is given by the formula $v = (1 - 2t)^2$.

a Find the acceleration of the car after t seconds.

b When does the car first stop?

c What is the acceleration at the instant when the car stops?

4 The displacement of a particle is given by the formula $s = \frac{t \sin t}{t - 1}$ $(t \in \mathbb{R}, t \neq 1)$. Find:

a a general formula for the velocity v

b the velocity when $t = 2$ secs

c a general formula for the acceleration a.

5 If the velocity of a particle is inversely proportional to the square root of the displacement travelled, prove that the acceleration is inversely proportional to the square of the displacement.

6 If the velocity of a particle is given by $v = e^{2s} \cos 2s$, show that the acceleration is $e^{4s}(\cos 4s - \sin 4s + 1)$.

7 A particle is moving along a straight line such that its displacement at any time t is given by the formula $s = 2 \cos 2t + 6 \sin 2t$.

a Show that the acceleration is directly proportional to the displacement.

b Using the compound angle formula $R \cos(2t + \alpha)$, where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$, show that the velocity is periodic and find the period.

8 The displacement of a particle is given by the formula $s = \frac{e^{2t}}{t^2 - 1}$, $t \in \mathbb{R}, t \neq 1$.

Find:

a a formula for the velocity of the particle at any time t

b the velocity of the particle after 2 seconds

c a formula for the acceleration of the particle

d the acceleration after 2 seconds in terms of e.

9 The velocity of a particle is given by $v^2 = \frac{6s^2}{\sqrt{s^2 - 1}}$. Show that the acceleration of the particle is given by the formula $a = \frac{12s^3 - 12s + 1}{2(s^2 - 1)^{\frac{3}{2}}}$.

10 David is visiting the fairground and his favourite ride is the big wheel. At any time t his horizontal displacement is given by the formula $s = 3 \sin(kt + c)$, where k and c are constants.

a Find his horizontal velocity at any time t.

b Find a general formula for the time when he first reaches his maximum horizontal velocity.

c Given that he has a horizontal displacement of $\frac{3}{\sqrt{2}}$ m after 10 seconds and a horizontal velocity of $\frac{3k\sqrt{3}}{2}$ m s^{-1} after 15 seconds, find the value of the acceleration after 20 seconds.

11 For a rocket to leave the earth's atmosphere, its displacement from the earth's surface increases exponentially with respect to time and is given by the formula $s = te^{kt^2}$ (for $t > 0$). Find:

a the value of k, given that when $t = 10$ seconds, $s = 3000$ m

b a general formula for the velocity at any time t

c a general formula for the acceleration at any time t

d the time when numerically the acceleration is twice the velocity $(t > 0)$.

12 The displacement of the East African mosquito has been modelled as a formula related to time, which is $s = \ln\left(\frac{t^2}{t - 1}\right)$. However, this formula is not totally successful, and works only for certain values of t. The maximum value of t is 20 seconds. The minimum value of t is the minimum point of the curve.

a Sketch the curve on a calculator and find the minimum value of t.

b Find the velocity of the mosquito at any time t and state any restrictions on the time t.

c Find the acceleration of the mosquito at any time t, stating any restrictions on t.

d Find the velocity and acceleration of the mosquito after 10 seconds.

13 The displacement of a train at any time t is given by the formula $s^2 + 2st - 2t^2 = 4$. Find:

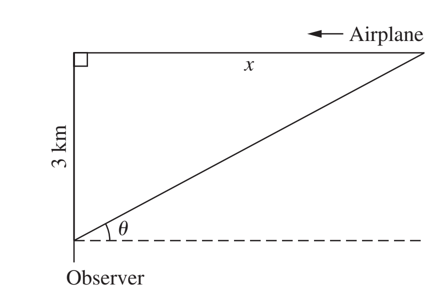
a the velocity in terms of s and t

b the acceleration in terms of s and t

c the relationship between the displacement and the time when the velocity has a stationary value.

Review exercise

1 An airplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level. At a given instant the observer notes that the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second. Find the speed, in kilometres per hour, at which the airplane is moving towards the observer. [IB Nov 03 P1 Q20]



2 Particle A moves in a straight line starting at O with a velocity in metres per second given by the formula $v_A = t^2 + 3t - 4$. Particle B also moves in a straight line starting at O with a velocity in metres per second given by the formula $v_B = 2te^{0.5t} - 3t^2$. Find:

a the acceleration of particle A when $t = 5$

b the times when the particles have the same velocity

c the maximum and minimum velocity of particle B in the range $0 \le t \le 3$.

3 Air is pumped into a spherical ball, which expands at a rate of 8 cm^3 per second $(8 \text{ cm}^3 \text{ s}^{-1})$. Find the exact rate of increase of the radius of the ball when the radius is 2 cm. [IB Nov 02 P1 Q16]

4 For a regular hexagon of side a cm and a circle of radius b cm, the sum of the perimeter of the hexagon and the circumference of the circle is 300 cm. What are the values of a and b if the sum of the areas is a minimum?

5 An astronaut on the moon throws a ball vertically upwards. The height, s metres, of the ball after t seconds is given by the equation $s = 40t + 0.5at^2$, where a is a constant. If the ball reaches its maximum height when $t = 25$, find the value of a. [IB May 01 P1 Q17]

6 A manufacturer of cans for Lite Lemonade needs to make cans that hold 500 ml of drink. A can is manufactured from a sheet of aluminium, and the area of aluminium used to make the cans needs to be a minimum.

a If the radius of the can is r and the height of the can is h, find an expression for the area A of aluminium needed to make one can.

b Hence find the radius of the can such that the surface area is a minimum.

c Find the surface area of this can.

7 A particle moves such that its displacement at any time t hours is given by the function $f(t) = 3t^2 \sin 5t, t > 0$. Find:

a the velocity at any time t

b the time when the particle first comes to rest

c the time when the particle first has its maximum velocity.

8 A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices are on the curve $y = \sin x$, where $0 \le x \le \pi$.

a Write down an expression for the area of the rectangle.

b Find the maximum area of the rectangle. [IB May 00 P1 Q17]

9 A triangle has two sides of length 5 cm and 8 cm. The angle θ between these two sides is changing at a rate of $\frac{\pi}{30}$ radians per minute. What is the rate of change of the area of the triangle when $\theta = \frac{\pi}{3}$?

10 A particle moves along a straight line. When it is a distance s from a fixed point O, where $s > 1$, the velocity v is given by $v = \frac{3s + 2}{2s - 1}$. Find the acceleration when $s = 2$. [IB May 99 P1 Q20]

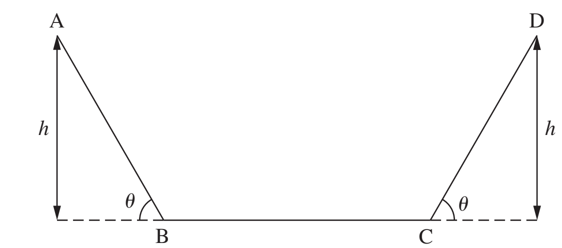
11 The depth h of the water at a certain point in the ocean at time t hours is given by the function $h = 2 \cos 3t - 3 \cos 2t + 6 \cos t + 15, t > 0$.

a Find the first time when the depth is a maximum.

b How long will it be before the water reaches its maximum depth again?

12 A square-based pyramid has a base of length x cm. The height of the pyramid is h cm. If the rate of change of x is 3 cm s^{-1}, and the rate of change of h is 2 cm s^{-1}, find the rate of change of the volume V when $x = 8$ cm and $h = 12$ cm.

13



A company makes channelling from a rectangular sheet of metal of width $2x$. A cross-section of a channel is shown in the diagram, where $AB + BC + CD = 2x$. The depth of the channel is h. AB and CD are inclined to the line BC at an angle θ. Find:

a the length of BC in terms of x, h and θ

b the area of the cross-section

c the maximum value of the cross-section as θ varies.

14 A drop of ink is placed on a piece of absorbent paper. The ink makes a cirular mark, which starts to increase in size. The radius of the circular mark is given by the formula $r = \frac{4(1 + t^4)}{8 + t^4}$, where r is the radius in centimetres of the circular mark and t is the time in minutes after the ink is placed on the paper.

a Find t when $r = \frac{17}{6}$.

b Find a simplified expression, in terms of t, for the rate of change of the radius.

c Find the rate of change of the area of the circular mark when $r = \frac{17}{6}$.

d Find the value of t when the rate of change of the radius starts to decrease, that is, find the value of t, $t > 0$, at the point of inflexion on the curve $r = \frac{4(1 + t^4)}{8 + t^4}$. [IB May 98 P2 Q2]