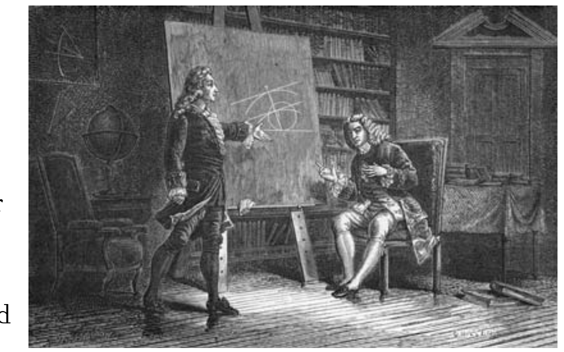
14 Integration 1



Jakob Bernoulli

Jakob Bernoulli was a Swiss mathematician born in Basel, Switzerland, on 27 December 1654. Along with his brother, Johann, he is considered to be one of the most important researchers of calculus after Newton and Leibniz. Jakob studied theology at university, but during this time he was studying mathematics and astronomy on the side, much against the wishes of his parents. After graduating in theology he travelled around Europe and worked with a number of the great mathematicians of the time. On return to Basel, it would have been natural for him to take an appointment in the church, but he followed his first love of mathematics and theoretical physics and took a job at the university. He was appointed professor of mathematics in 1687 and, along with his brother, Johann, started studying Leibniz's work on calculus. At this time Leibniz's theories were very new, and hence the work done by the two brothers was at the cutting edge. Jakob worked on a variety of mathematical ideas, but in 1690 he first used the term "integral" with the meaning it has today. Jakob held the chair of mathematics at the university in Basel until his death in 1705. Jakob had always been fascinated by the properties of the logarithmic spiral, and this was engraved on his tombstone along with the words "Eadem Mutata Resurgo" which translates as "I shall arise the same though changed."

14.1 Undoing differentiation

In Chapters 8–10 we studied differential calculus and saw that by using the techniques of differentiation the gradient of a function or the rate of change of a quantity can be found. If the rate of change is known and the original function needs to be found, it is necessary to "undo" differentiation. Integration is this "undoing", the reverse process to differentiation. Integration is also known as anti-differentiation, and this is often the best way of looking at it.

If $\dfrac{dy}{dx} = 2x$, what is the original function y?

This is asking what we started with in order to finish with a derived function of $2x$. Remembering that we differentiate by multiplying by the power and then subtracting one from the power, we must have started with x^2.

Similarly, if $\dfrac{dy}{dx} = 4$, then this must have started as $4x$.

Exercise 1

Find the original function.

1 $\frac{dy}{dx} = 5$ **2** $\frac{dy}{dx} = 10$ **3** $\frac{dy}{dx} = -2$ **4** $\frac{dy}{dx} = 4x$

5 $\frac{dy}{dx} = 12x$ **6** $\frac{dy}{dx} = 3x^2$ **7** $\frac{dy}{dx} = 4x^3$ **8** $\frac{dy}{dx} = 5x^4$

9 $\frac{dy}{dx} = 9x^2$ **10** $\frac{dy}{dx} = -4x^{-2}$

Looking at the answers to Exercise 1, we can form a rule for anti-differentiation.

If we describe the process of "undoing" differentiation for this type of function, we could say "add 1 to the power and divide by that new power".

In mathematical notation this is

This symbol means "the integral of".

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

This means "with respect to x" – the variable we are concerned with.

14.2 Constant of integration

Again consider the situation of $\frac{dy}{dx} = 4$. Geometrically, this means that the gradient of the original function, y, is constant and equal to 4.

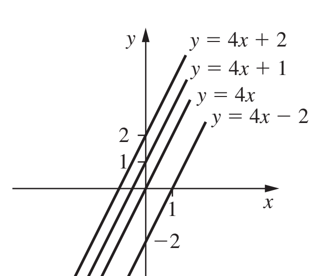
So $y = 4x + c$ (c is the y-intercept of the line from the general equation of a line $y = mx + c$).

Consider the lines $y = 4x - 3$, $y = 4x$ and $y = 4x + 5$.

For each line $\frac{dy}{dx} = 4$ (as the gradient is 4 each time). Remember that, when differentiating, a constant "disappears" because the gradient of a horizontal line is zero or alternatively the derivative of a constant is zero.

So $\int 4\,dx = 4x + c$.

Unless more information is given (a point on the line), then the value of c remains unknown. So any of the lines below could be the original function with $\frac{dy}{dx} = 4$. In fact there are an infinite number of lines that could have been the original function.



This "c" is called the **constant of integration**. It must be included in the answer of any integral, as we do not know what constant may have "disappeared" when the function was differentiated.

Example

Find y if $\frac{dy}{dx} = 8x$.

$y = \int 8x\,dx = 4x^2 + c$

Example

Find $\int 10x - 7\,dx$.

So $\int 10x - 7\,dx = \int 10x\,dx + \int -7\,dx$

So $\int 10x - 7\,dx = 5x^2 - 7x + c$

We established in Chapter 8 that a function could be differentiated "term by term". In a similar way, this can be integrated term by term.

Example

Integrate $8x^{-2} + x^{\frac{3}{4}}$.

$\int 8x^{-2} + x^{\frac{3}{4}}\,dx = -8x^{-1} + \frac{4}{7}x^{\frac{7}{4}} + c$

The coefficient of a term has no effect on the process of integration, and so a constant can be "taken out" of the integral. This is demonstrated in the next example.

Example

Find the solution of $\frac{dy}{dx} = \frac{2}{9}x^{-\frac{1}{3}}$.

This is asking to find y by integrating.

So

$y = \int \frac{2}{9}x^{-\frac{1}{3}}\,dx$

$y = \frac{2}{9}\int x^{-\frac{1}{3}}\,dx$

$y = \frac{2}{9}\cdot\frac{3}{2}x^{\frac{2}{3}} + c$

$y = \frac{1}{3}x^{\frac{2}{3}} + c$

As with the expressions differentiated in Chapters 8–10, it is sometimes necessary to simplify the function prior to integrating.

Example

Find $\int \frac{x^2 - x^5}{3\sqrt{x}} dx$.

$$\int \frac{x^2 - x^5}{3\sqrt{x}} dx$$
$$= \frac{1}{3}\int x^{-\frac{1}{2}}(x^2 - x^5)\ dx$$
$$= \frac{1}{3}\int x^{\frac{3}{2}} - x^{\frac{9}{2}}\ dx$$
$$= \frac{1}{3}\left[\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{11}x^{\frac{11}{2}}\right] + c$$
$$= \frac{2}{15}x^{\frac{5}{2}} - \frac{2}{33}x^{\frac{11}{2}} + c$$

The integral sign and the dx remain until the integration is performed.

The c can remain outside all brackets as it is an arbitrary constant and so, when multiplied by another constant, it is still a constant.

Example

Find $\int \frac{4}{p^2} dp$.

$$\int \frac{4}{p^2} dp$$
$$= \int 4p^{-2}\ dp$$
$$= -4p^{-1} + c$$
$$= -\frac{4}{p} + c$$

Here p is the variable and so the integral is with respect to p.

Exercise 2

Integrate these expressions.

1 $2x - 1$ **2** x^2 **3** x^3 **4** x^4

5 $6x^2 - 5$ **6** $8x^3 + 4x - 3$ **7** $5x^2 - 4$ **8** x^{-2}

9 $x^{\frac{1}{2}}$ **10** $x^{-\frac{2}{3}}$ **11** $7 - 4x^{-3}$

Find these integrals.

12 $\int x^{-\frac{1}{2}}\ dx$ **13** $\int 2x^6 - 5x^4\ dx$ **14** $\int 5x^{\frac{3}{2}} - 4x^{-3}\ dx$

15 $\int 4x^3 + 4x - 9\ dx$ **16** $\int 1 - 2x + 6x^2 - x^3\ dx$ **17** $\int \frac{6}{x^3} dx$

Find the solution of these.

18 $\frac{dy}{dx} = \frac{2}{x^5}$ **19** $\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}}$ **20** $\frac{dy}{dx} = 8 - \frac{3}{x^{\frac{1}{3}}}$

21 $\frac{dy}{dx} = 8x(2x^2 - 3)$ **22** $\frac{dy}{dx} = (x - 9)(2x - 3)$ **23** $\frac{dy}{dx} = (3x - 4)^2$

24 $\frac{dy}{dx} = \frac{x^2 - 5}{x^5}$ **25** $\frac{dy}{dx} = \frac{4x^3 - 7x}{\sqrt{x}}$ **26** $\frac{dy}{dx} = \frac{7x^4 - 6x^{\frac{3}{4}}}{2x^{\frac{1}{4}}}$

Find y by integrating with respect to the relevant variable.

27 $\frac{dy}{dp} = \frac{12}{p^3}$ **28** $\frac{dy}{dk} = 8k^{\frac{5}{4}}$ **29** $\frac{dy}{dz} = z^3\left(z^2 - \frac{1}{z^2}\right)$

30 $\frac{dy}{dt} = \frac{(t + 3)^2}{t^4}$ **31** $\frac{dy}{dt} = \frac{\sqrt{t} - 4t^3}{3t}$

14.3 Initial conditions

In all of the integrals met so far, it was necessary to include the constant of integration, c. However, if more information is given (often known as initial conditions), then the value of c can be found.

Consider again the example of $\frac{dy}{dx} = 4$. If the line passes through the point (1, 3) then c can be evaluated and the specific line found.

So $\frac{dy}{dx} = 4$

$\Rightarrow y = \int 4\ dx$

$\Rightarrow y = 4x + c$

Since we know that when $x = 1, y = 3$ these values can be substituted into the equation of the line.

So $3 = 4 \times 1 + c$

$\Rightarrow c = -1$

The equation of the line is $y = 4x - 1$.

When the value of c is unknown, this is called the **general solution**.

If the value of c can be found, this is known as the **particular solution**.

Example

Given that the curve passes through (2, 3) and $\frac{dy}{dx} = 2x - 1$, find the equation of the curve.

$\frac{dy}{dx} = 2x - 1$

$\Rightarrow y = \int 2x - 1\ dx$

$\Rightarrow y = x^2 - x + c$

Using (2, 3) $\Rightarrow 3 = 1^2 - 1 + c$

$\Rightarrow c = 3$

So the equation of the curve is $y = x^2 - x + 3$.

Example

Find P given that $\frac{dP}{dt} = \frac{1}{\sqrt{t}}$ and $P = 7$ when $t = 100$

$$\frac{dP}{dt} = \frac{1}{\sqrt{t}}$$

$$\Rightarrow P = \int \frac{1}{\sqrt{t}} dt$$

$$\Rightarrow P = \int t^{-\frac{1}{2}} \, dt$$

$$\Rightarrow P = 2t^{\frac{1}{2}} + c$$

Since $P = 7$ when $t = 100$, $7 = 2\cdot\sqrt{100} + c$

$$\Rightarrow c = -13$$

Hence $P = 2t^{\frac{1}{2}} - 23$

Exercise 3

Given the gradient of each curve, and a point on that curve, find the equation of the curve.

1 $\frac{dy}{dx} = 6, (2, 8)$

2 $\frac{dy}{dx} = 4x, (1, 5)$

3 $\frac{dy}{dx} = 8x - 3, (-2, 4)$

4 $\frac{dy}{dx} = -2x + 5, (4, 4)$

5 $\frac{dy}{dx} = 4x^3 - 6x^2 + 7, (1, 9)$

6 $\frac{dy}{dx} = 4x^2 + \frac{6}{x^2}, (4, -1)$

7 $\frac{dy}{dx} = \frac{8}{\sqrt{x}}, (9, 2)$

Find the particular solution, using the information given.

8 $\frac{dy}{dt} = t^2(t^4 - 3t^2 - 4), y = 6$ when $t = 1$

9 $\frac{dQ}{dp} = \frac{p^3 - 4p^5}{3\sqrt{p}}, Q = 2$ when $p = 0$

14.4 Basic results

Considering the basic results from differentiation, standard results for integration can now be produced. For polynomials, the general rule is:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

However, consider $\int \frac{1}{x} dx = \int x^{-1} \, dx$.

Using the above rule, we would obtain $\frac{x^0}{0}$, but this is not defined. However, it is known that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

This provides the result that $\int \frac{1}{x} dx = \ln|x| + c$

Remembering that $\ln x$ is defined only for positive values of x, we recognize that $\int \frac{1}{x} dx = \ln|x| + c$, taking the absolute value of x. As $\frac{1}{x}$ is defined for all $x \in \mathbb{R}, x \neq 0$ and $\ln x$ is defined only for $x > 0$, the absolute value sign is needed so that we can integrate $\frac{1}{x}$ for all values of x for which it is defined.

Similarly $\frac{d}{dx}(e^x) = e^x$ so $\int e^x \, dx = e^x + c$

When differentiating sine and cosine functions the following diagram was used and is now extended:

Differentiate ↓	S C −S −C	↑ Integrate

The integrals of other trigonometric functions can be found by reversing the basic rules for differentiation, and will be discussed further in Chapter 15.

Standard results

Function	Integral + c
$f(x)$	$\int f(x) \, dx$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln x$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$

Example

Integrate $\sin x - e^x$.

$$\int \sin x - e^x \, dx$$

$$= -\cos x - e^x + c$$

Example

Integrate $\dfrac{3}{x} + 4 \cos x$.

$$\int \frac{3}{x} + 4 \cos x \, dx$$

$$= 3 \ln|x| + 4 \sin x + c$$

Exercise 4

Integrate these functions.

1 $x^3 - \dfrac{2}{x}$ **2** $4e^x + \sin x$ **3** $\dfrac{5}{x} - \cos x$

4 $6 \sin x - 6x^4$ **5** $-8 \sin x + 7e^x$ **6** $5e^x - 2 \sin x + \dfrac{3}{x}$

7 $\dfrac{e^x}{3} - \dfrac{5}{2x} + 7 \sin x$ **8** $\dfrac{e^x}{15} - 15\sqrt{x} + \cos x$

14.5 Anti-chain rule

When functions of the type $(2x - 1)^5$, e^{8x} and $\sin\left(2x - \dfrac{\pi}{3}\right)$ are differentiated, the chain rule is applied. The chain rule states that we multiply the derivative of the outside function by the derivative of the inside function. So to integrate functions of these types we consider what we started with to obtain that derivative.

Example

$\dfrac{dy}{dx} = (2x - 1)^3$. Find y.

So $y = \displaystyle\int (2x - 1)^3 \, dx$

When integrating, 1 is added to the power, so y must be connected to $(2x - 1)^4$. Since we multiply by the power and by the derivative of the inside function when differentiating, we need to balance this when finding y.

So $y = \dfrac{1}{4} \cdot \dfrac{1}{2}(2x - 1)^4 + c$

$\Rightarrow y = \dfrac{1}{8}(2x - 1)^4 + c$

Example

Find $\displaystyle\int \cos 3x \, dx$.

Using Diff $\downarrow \begin{array}{c} S \\ C \\ -S \\ -C \end{array} \uparrow$ Int this begins with $\sin 3x$.

Balancing to obtain $\cos 3x$ when differentiating

$\Rightarrow \displaystyle\int \cos 3x \, dx = \frac{1}{3} \sin 3x + c$

Example

Find $\displaystyle\int 6e^{4x} \, dx$.

This started with e^{4x} as $\dfrac{d}{dx}(e^{4x}) = 4e^{4x}$.

So $\displaystyle\int 6e^{4x} \, dx = 6\int e^{4x} \, dx$

$= 6 \cdot \dfrac{1}{4} e^{4x} + c$

$= \dfrac{3}{2} e^{4x} + c$

Example

Find y given that $\dfrac{dy}{dx} = \dfrac{1}{3x - 4} - \sin\left(4x - \dfrac{\pi}{2}\right)$.

So $y = \displaystyle\int \frac{1}{3x - 4} - \sin\left(4x - \frac{\pi}{2}\right) dx$

As $\dfrac{1}{3x - 4} = (3x - 4)^{-1}$

we recognize that this comes from $\ln|3x - 4|$ as $\dfrac{d}{dx}(\ln|3x - 4|) = \dfrac{3}{3x - 4}$.

So $y = \dfrac{1}{3} \ln|3x - 4| + \dfrac{1}{4} \cos\left(4x - \dfrac{\pi}{2}\right) + c$

For these simple cases of the "anti-chain rule", we divide by the derivative of the inside function each time. With more complicated integrals, which will be met in the next chapter, this is not always the case, and at that point the results will be formalized. This is why it is useful to consider these integrals as the reverse of differentiation.

Exercise 5

Find these integrals.

1 $\int \sin 5x\,dx$ **2** $\int \cos 6x\,dx$ **3** $\int \sin 2x\,dx$ **4** $\int \sin\frac{1}{2}x\,dx$

5 $\int 8\cos 4x\,dx$ **6** $\int -6\sin 3x\,dx$ **7** $\int -5\cos 2x\,dx$ **8** $\int e^{6x}\,dx$

9 $\int e^{5x}\,dx$ **10** $\int 4e^{4x}\,dx$ **11** $\int 8e^{6t}\,dt$ **12** $\int -5e^{6p}\,dp$

13 $\int 8x - e^{2x}\,dx$ **14** $\int 4e^{-2x}\,dx$

Find y if:

15 $\frac{dy}{dx} = \frac{1}{2x-3}$ **16** $\frac{dy}{dx} = \frac{1}{8x+7}$ **17** $\frac{dy}{dx} = \frac{4}{2x-5}$

18 $\frac{dy}{dx} = (3x-1)^5$ **19** $\frac{dy}{dx} = (4x-7)^6$ **20** $\frac{dy}{dx} = (4x+3)^{-3}$

21 $\frac{dy}{dx} = (3-2x)^4$ **22** $\frac{dy}{dx} = \frac{4}{(3x-2)^3}$ **23** $\frac{dy}{dt} = \frac{3}{(2t-1)^2}$

24 $\frac{dy}{dx} = \frac{4}{3x-1}$ **25** $\frac{dy}{dx} = \frac{6}{3x-5}$ **26** $\frac{dy}{dp} = \frac{8}{4-p}$

27 $\frac{dy}{dt} = \frac{3}{6-t}$

Integrate these functions.

28 $6e^{4x}$ **29** $\sin 3x - 4x$ **30** $4e^{-8x} - 4\cos 2x$

31 $\frac{1}{2x-1} + (3x+4)^5$ **32** $6x^2 - \frac{2}{3x+2}$

14.6 Definite integration

Definite integration is where the integration is performed between limits, and this produces a numerical answer.

A definite integral is of the form $\int_a^b f(x)\,dx$

Upper limit $(x = b)$

Lower limit $(x = a)$

When a definite integral is created, the lower limit is always smaller than the upper limit.

Example

$$\int_2^3 2x - 1\,dx$$

$$= \left[x^2 - x\right]_2^3$$

This notation means that the integration has taken place. The two values are now substituted into the function and subtracted.

$$= (3^2 - 3) - (2^2 - 2)$$
$$= 6 - 2$$
$$= 4$$

There is no constant of integration used here. This is because it cancels itself out and so does not need to be included.

$$\left[x^2 - x + c\right]_2^3$$
$$= (3^2 - 3 + c) - (2^2 - 2 + c)$$
$$= 6 + c - 2 - c$$
$$= 4$$

Example

$$\int_0^4 (2x-3)^2\,dx$$

$$= \left[\frac{1}{6}(2x-3)^3\right]_0^4$$

$$= \frac{1}{6}(8-3)^3 - \left(\frac{1}{6}(0-3)^3\right)$$

$$= \frac{125}{6} + \frac{27}{6}$$

$$= \frac{152}{6}$$

$$= \frac{76}{3}$$

Example

$$\int_0^{\frac{\pi}{4}} \sin 2x + 1\,dx$$

$$= \left[-\frac{1}{2}\cos 2x + x\right]_0^{\frac{\pi}{4}}$$

$$= \left(-\frac{1}{2}\cos\frac{\pi}{2} + \frac{\pi}{4}\right) - \left(-\frac{1}{2}\cos 0 + 0\right)$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

$$= \frac{\pi + 2}{4}$$

To differentiate trigonometric functions, we always use radians, and the same is true in integration.

Example

$$\int_{-2}^{2} \frac{2}{x-4}\,dx$$

$= \left[2\ln|x-4|\right]_{-2}^{2}$

$= 2\ln|-2| - 2\ln|-6|$

$= 2\ln 2 - 2\ln 6$

$= \ln 4 - \ln 36$

$= \ln\frac{4}{36}$

$= \ln\frac{1}{9}$

$= -2.20$

An answer could have been approximated earlier, but if an exact answer were required, this form would need to be given.

Example

$$\int_{2}^{a} e^{4x}\,dx \quad \text{where } a > 2$$

$= \left[\frac{1}{4}e^{4x}\right]_{2}^{a}$

$= \frac{1}{4}e^{4a} - \frac{1}{4}e^{8}$

$= \frac{1}{4}(e^{4a} - e^{8})$

Although there is no value for the upper limit, an answer can still be found that is an expression in a.

Exercise 6

Find the value of these definite integrals

1 $\int_{1}^{2} 2x\,dx$

2 $\int_{2}^{3} 6x^2\,dx$

3 $\int_{0}^{4} 5\,dx$

4 $\int_{0}^{2} 8x - 4x^3\,dx$

5 $\int_{2}^{3} \frac{4}{x^2}\,dx$

6 $\int_{-3}^{-1} \frac{6}{x^3}\,dx$

7 $\int_{0}^{3} e^{2x}\,dx$

8 $\int_{2}^{4} 4e^{3x}\,dx$

9 $\int_{-2}^{1} 2e^{-4x}\,dx$

10 $\int_{0}^{\frac{\pi}{3}} \sin 3x\,dx$

11 $\int_{0}^{\frac{\pi}{2}} \cos 2\theta\,d\theta$

12 $\int_{0}^{\frac{\pi}{6}} \cos 3t - 6\,dt$

13 $\int_{0}^{3} (2x-1)^2\,dx$

14 $\int_{-2}^{0} (3x-1)^3\,dx$

15 $\int_{-1}^{2} (3-2x)^4\,dx$

16 $\int_{1}^{4} \frac{2}{(2x-1)^3}\,dx$

17 $\int_{-2}^{-1} \frac{3}{(3x-4)^2}\,dx$

18 $\int_{2}^{5} \frac{1}{2x-1}\,dx$

19 $\int_{2}^{4} \frac{4}{3x-4}\,dx$

20 $\int_{-4}^{-1} \frac{5}{t-2}\,dt$

21 $\int_{1}^{2} 6p - \frac{3}{4p+1}\,dp$

22 $\int_{-p}^{p} \sin 4x - 6x\,dx$

23 $\int_{0}^{k} \frac{4}{2x+1}\,dx$

14.7 Geometric significance of integration

When we met differentiation, it was considered as a technique for finding the gradient of a function at any point. We now consider the geometric significance of integration.

Consider $\int_{1}^{5} 3\,dx$.

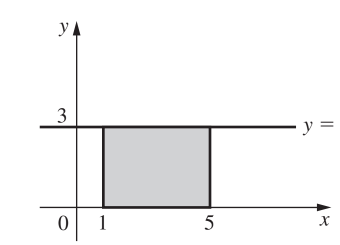
Remember the limits are values of x.

$$\int_{1}^{5} 3\,dx$$

$= \left[3x\right]_{1}^{5}$

$= 15 - 3$

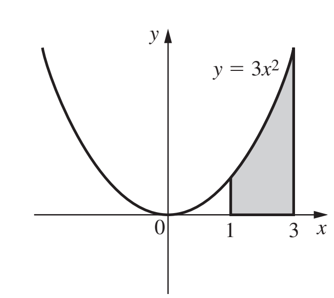
$= 12$



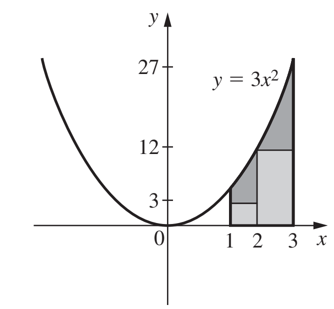
This is the same as the area enclosed by the function, the x-axis and the vertical lines $x = 1$ and $x = 5$.

This suggests that the geometric interpretation of integration is **the area between the curve and the x-axis.**

Consider $y = 3x^2$.



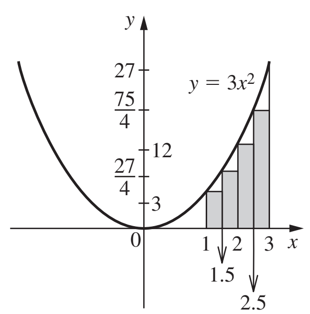
In the previous example where $y = 3$, it was easy to find the area enclosed by the function, the x-axis and the limits. However, to find the area under a curve is less obvious. This area could be approximated by splitting it into rectangles.



$$A \simeq 1 \times 3 + 1 \times 12$$
$$= 15 \text{ square units}$$

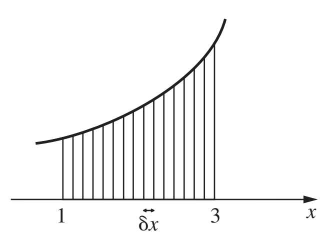
This is clearly not a very accurate approximation, so to make it more accurate thinner rectangles are used.

Four rectangles would make this more accurate.



$$A \simeq \frac{1}{2} \times 3 + \frac{1}{2} \times \frac{27}{4} + \frac{1}{2} \times 12 + \frac{1}{2} \times \frac{75}{4}$$
$$A \simeq \frac{81}{4}$$

As the rectangles become thinner, the approximation becomes more accurate.



This can be considered in a more formal way. Each strip has width δx with height y and area δA.

So $\delta A \simeq y \cdot \delta x$

$\Rightarrow A = \sum \delta A$

$\Rightarrow A \simeq \sum y\, \delta x$

As δx gets smaller, the approximation improves

and so $A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y \delta x.$

Now $y \simeq \dfrac{\delta A}{\delta x}$

$\Rightarrow y = \lim_{\delta x \to 0} \dfrac{\delta A}{\delta x}$

$\Rightarrow y = \dfrac{dA}{dx}$

Integrating gives $\int y\, dx = \int \frac{dA}{dx} dx = \int dA = A + c$

Limits are needed here to specify the boundaries of the area.

With the boundary conditions,

c can now be ignored.

$$A = \int_a^b y\, dx$$

This is the basic formula for finding the area between the curve and the x-axis.

Hence $\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y \cdot \delta x = \int_a^b y\, dx.$

So the $\int$ sign actually means "sum of" (it is an elongated S).

This now shows more formally that the geometric significance of integration is that it finds the area between the curve and the x-axis.

If the two notations for summation are compared, we find that sigma notation is used for a discrete variable and that integral notation is used for a continuous variable.

$\sum_1^3 3x^2$ and $\int_1^3 3x^2\, dx$

This is a sum of a discrete variable.

This is the sum of a continuous variable.

$= \left[x^3\right]_1^3$

$= 27 - 1$

$= 26$

The area required is 26 square units.

Example

Find the area given by $\int_2^4 2x - 1\, dx$.

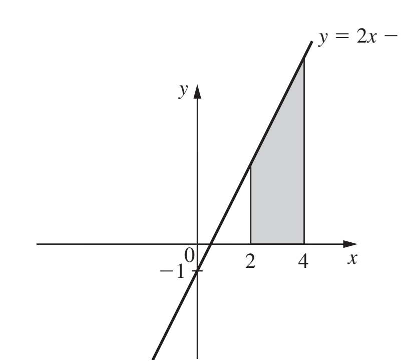
$\int_2^4 2x - 1\, dx$

$= \left[x^2 - x\right]_2^4$

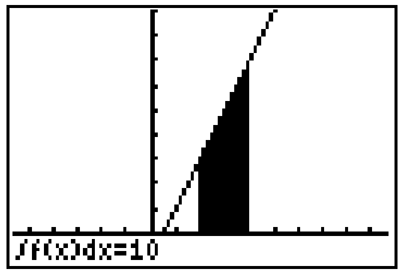
$= (16 - 4) - (4 - 2)$

$= 12 - 2$

$= 10$ square units



This integration can be performed on a calculator. Although a calculator cannot perform calculus algebraically, it can calculate definite integrals. This is shown in the diagram below, and we find the area is still 10 square units.



Some definite integration questions require the use of a calculator. Where an exact value is required, one of the limits is a variable or the question is in a non-calculator paper, algebraic methods must be employed.

Example

Find the area enclosed by $y = \cos 2x$ and the x axis, between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

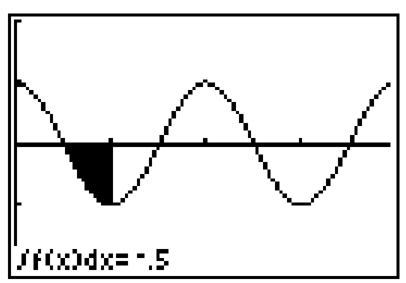
$$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2x \, dx$$

$$= \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\sin \pi - \frac{1}{2}\sin\frac{\pi}{2}$$

$$= 0 - \frac{1}{2}$$

$$= -\frac{1}{2}$$



The answer to this definite integral is negative. However, area is a scalar quantity (it has no direction, only magnitude) and so the area required is $\frac{1}{2}$ square unit. As the negative sign has no effect on the area, the area can be considered to be the absolute value of the integral. So $A = \left|\int_a^b f(x)\, dx\right|$. Whether the calculation is done using the absolute value sign or whether we do the calculation and then ignore the negative sign at the end does not matter. As can be seen from the graph, the significance of the negative sign is that the area is contained below the x-axis.

Example

Find the area given by $\int_{-2}^{-1} \frac{1}{x} dx$.

$$A = \left|\int_{-2}^{-1} \frac{1}{x} dx\right|$$

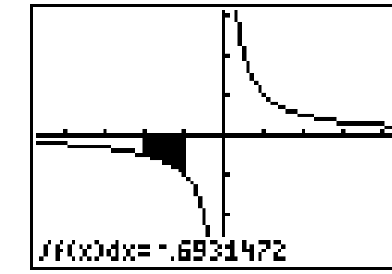
$$= \left|\Big[\ln|x|\Big]_{-2}^{-1}\right|$$

$$= \big|\ln|-1| - \ln|-2|\big|$$

$$= |\ln 1 - \ln 2|$$

$$= |-\ln 2|$$

$$= 0.693$$



So the required area is 0.693 square units.

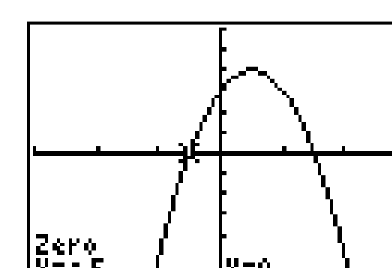
It should be noted that it is actually not possible to find the area given by $\int_{-2}^{1} \frac{1}{x} dx$ as there is a vertical asymptote at $x = 0$. It is not possible to find the definite integral over an asymptote of any curve, as technically the area would be infinite.

This example also provides another explanation for the need for the modulus signs in $\int \frac{1}{x} dx = \ln|x|$. Although logarithms are not defined for negative values of x, in order to find the area under a hyperbola like $y = \frac{1}{x}$, which clearly exists, negative values need to be substituted into a logarithm, and hence the absolute value is required. This was shown in the above example.

Example

Find the area enclosed by $y = -(2x - 1)^2 + 4$ and the x-axis.

First the limits need to be found.



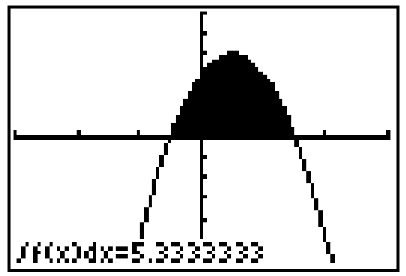
These are the roots of the graph (they can be found algebraically or by using a calculator)

$-(2x-1)^2+4=0$

$\Rightarrow 2x-1=\pm 2$

$\Rightarrow x=-\frac{1}{2}$ or $x=\frac{3}{2}$

So the area is given by



$$\int_{-\frac{1}{2}}^{\frac{3}{2}} -(2x-1)^2+4\,dx$$

$$=\left[-\frac{1}{6}(2x-1)^3+4x\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$=\frac{16}{3}$$

Exercise 7

Find the area given by these definite integrals.

1 $\int_2^6 8\,dx$ **2** $\int_0^3 2x+3\,dx$ **3** $\int_1^3 6x^2-1\,dx$

4 $\int_0^2 (3x-2)^4\,dx$ **5** $\int_0^{\pi} \sin x\,dx$ **6** $\int_0^2 e^x\,dx$

7 $\int_1^4 \frac{4}{x}dx$ **8** $\int_2^5 \frac{2}{2x+3}dx$ **9** $\int_{-4}^{-2} 3x-2\,dx$

10 $\int_{-4}^{-2} \frac{4}{2x+1}dx$ **11** $\int_0^1 e^{2x-1}-4x\,dx$ **12** $\int_0^{\frac{\pi}{3}} \cos 3x+4x\,dx$

Find the area enclosed by the curve and the x-axis.

13 $y=x^2+2$ and the lines $x=1$ and $x=4$

14 $y=e^{6x}$ and the lines $x=-2$ and $x=1$

15 $y=\sin x$ and the lines $x=0$ and $x=\pi$

16 $y=\frac{4}{3x+4}$ and the lines $x=-5$ and $x=-2$

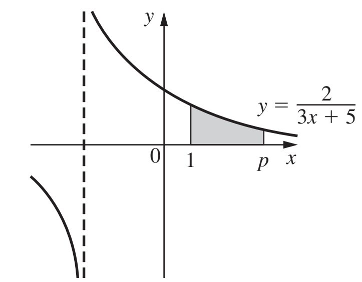
17 $y=1-x^2$ and the lines $x=-1$ and $x=1$

18 $y=-(4x+1)^2+9$ and the lines $x=-1$ and $x=\frac{1}{2}$

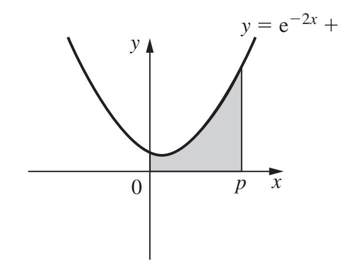
19 $y=x^3-2x^2$ and the lines $x=0$ and $x=2$

20 $y=e^{2x}-\sin 2x$ and the line $x=-\frac{\pi}{2}$ and the y-axis

21 Find an expression in terms of p for this area.



22 Find an expression in terms of p for this area.



23 Find $k\,(k>0)$ given that $\int_0^k 3x^{\frac{1}{2}}\,dx=16$.

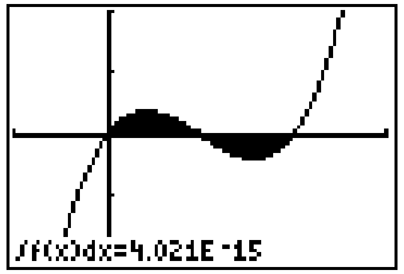
24 Find a given that $\int_{-a}^{a} \frac{25}{(9-x)^2}dx=\frac{5}{8}$.

14.8 Areas above and below the x-axis

In this case the formula needs to be applied carefully. Consider $y=x(x-1)(x-2)$ and the area enclosed by this curve and the x-axis.

$y=x^3-3x^2+2x$

If the definite integral $\int_0^2 x^3-3x^2+2x\,dx$ is calculated, we obtain an answer of 0 (remember the calculator uses a numerical process to calculate an integral) and so this result is interpreted as zero.



However, it is clear that the area is not zero.

We know that a definite integral for an area below the x-axis provides a negative answer. This explains the zero answer – the two (identical) areas have cancelled each other out. So although the answer to the definite integral is zero, the area is not zero.

To find an area that has parts above and below the x-axis, consider the parts separately.

So in the above example, area $= 2 \times \int_0^1 x^3 - 3x^2 + 2x \, dx$

$$= 2 \times \frac{1}{4}$$

$$= \frac{1}{2} \text{ unit}^2$$

This demonstrates an important point. The answer to finding an area and to finding the value of the definite integral may actually be different.

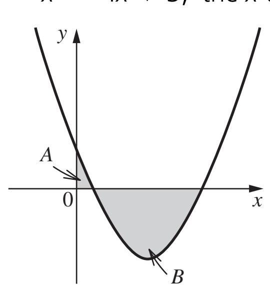
The method used in the example below shows how to avoid such problems.

Method

1 Sketch the curve to find the relevant roots of the graph.
2 Calculate the areas above and below the x-axis separately.
3 Add together the areas (ignoring the negative sign).

Example

Find the area enclosed by $y = x^2 - 4x + 3$, the x-axis and the y-axis.



1. Sketch the curve and shade the areas required.

The roots of the graph are given by $x^2 - 4x + 3 = 0$

$$\Rightarrow (x - 1)(x - 3) = 0$$

$$\Rightarrow x = 1 \text{ or } x = 3$$

These results can be found using a calculator.

2. Work out the areas separately.

$$A = \left| \int_0^1 x^2 - 4x + 3 \, dx \right| \qquad B = \left| \int_1^3 x^2 - 4x + 3 \, dx \right|$$

$$= \left| \left[\frac{1}{3}x^3 - 2x^2 + 3x \right]_0^1 \right| \qquad = \left| \left[\frac{1}{3}x^3 - 2x^2 + 3x \right]_1^3 \right|$$

$$= \left| \left(\frac{1}{3} - 2 + 3 \right) - (0) \right| \qquad = \left| (9 - 18 + 9) - \left(\frac{4}{3} \right) \right|$$

$$= \frac{4}{3} \qquad = \left| -\frac{4}{3} \right| = \frac{4}{3}$$

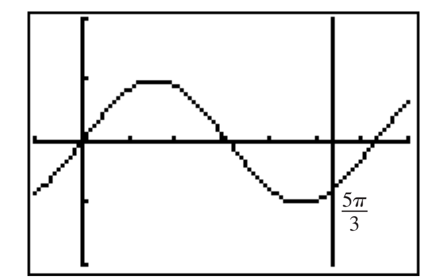
3. These areas can be calculated on a calculator (separately) and then added.

So the total area $= \frac{8}{3}$ square units.

Example

Find the area bounded by $y = \sin x$, the y-axis and the line $x = \frac{5\pi}{3}$.

Graphing this on a calculator,



This area can be split into:

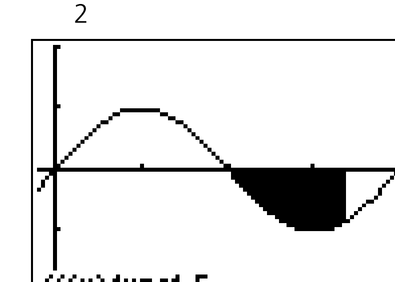
$$\int_0^{\pi} \sin x \, dx \qquad \text{and} \qquad \int_{\pi}^{\frac{5\pi}{3}} \sin x \, dx$$

$$= \left[-\cos x \right]_0^{\pi} \qquad = \left[-\cos x \right]_{\pi}^{\frac{5\pi}{3}}$$

$$= (-\cos \pi) - (-\cos 0) \qquad = \left(-\cos \frac{5\pi}{3} \right) - (-\cos \pi)$$

$$= 1 - (-1) \qquad = \left(-\frac{1}{2} \right) - (1)$$

$$= 2 \qquad = -\frac{3}{2}$$

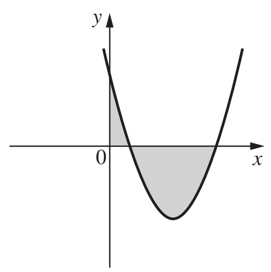


So the total area is $\frac{7}{2}$ square units.

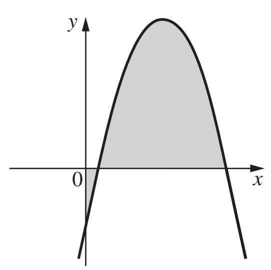
Exercise 8

Find the shaded area on the following diagrams.

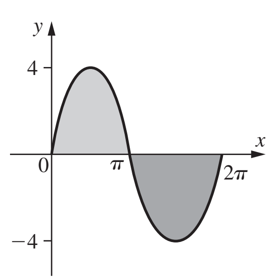
1 $y = x^2 - 8x + 12$



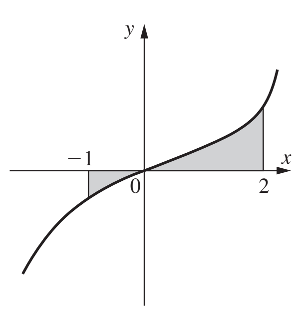
2 $y = -x^2 + 9x - 8$



3 $y = 4 \sin x$



4 $y = 5x^3$



Find the area bounded by the curve and the x-axis in the following cases.

5 $y = x(x + 3)(x - 2)$
6 $y = -(x + 4)(2x + 1)(x - 3)$
7 $y = x^3 - x^2 - 16x + 16$
8 $y = 6x^3 - 5x^2 - 12x - 4$

Find the area bounded by the curve, the x-axis and the lines given.

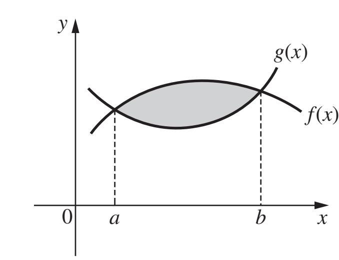
9 $y = x^2 - x - 6$, $x = -2$ and $x = 4$
10 $y = 6x^3$, $x = -4$ and $x = 2$
11 $y = 3 \sin 2x$, $x = 0$ and $x = \frac{5\pi}{6}$
12 $y = \cos\left(2x - \frac{\pi}{6}\right)$, $x = 0$ and $x = \pi$
13 $y = 4 \cos 2x$, $x = \frac{\pi}{2}$ and $x = \frac{5\pi}{4}$
14 $y = x^3 - e^x$, $x = 0$ and $x = 4$

14.9 Area between two curves

The area contained between two curves can be found as follows.

The area under f(x) is given by $\int_a^b f(x)\,dx$ and under g(x) is given by $\int_a^b g(x)\,dx$.

So the shaded area is $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.



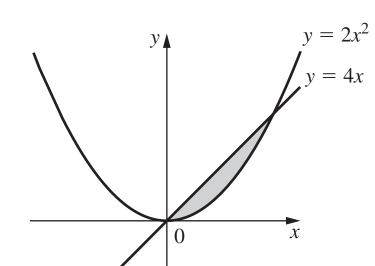
Combining these gives $\int_a^b f(x) - g(x)\,dx$.

This can be expressed as $\int_a^b$ upper curve – lower curve dx.

As long as we always take upper – lower, the answer is positive and hence it is not necessary to worry about above and below the x-axis.

Example

Find the shaded area.



The functions intersect where $2x^2 = 4x$.

$$\Rightarrow 2x^2 - 4x = 0$$
$$\Rightarrow 2x(x - 2) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 2$$

These intersection points can also be found using a calculator.

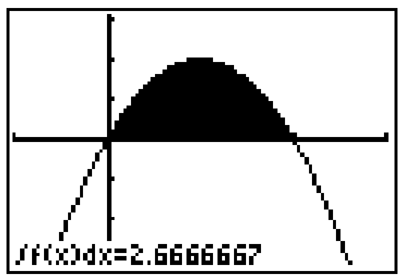
So the area $= \int_0^2 4x - 2x^2 \, dx$

$$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2$$

$$= \left(8 - \frac{16}{3}\right) - (0)$$

$$= \frac{8}{3}$$

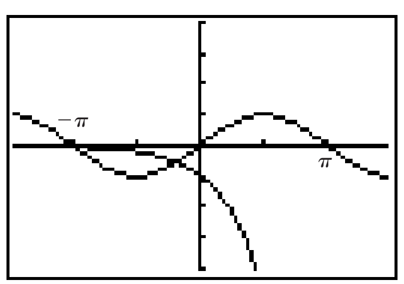
This function can be drawn using Y1-Y2 and then the area calculated.



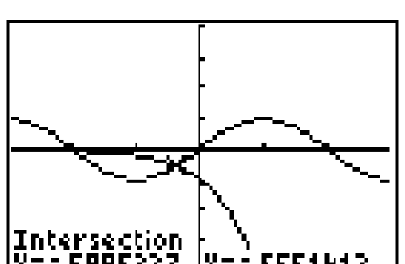
The two limits are roots of the resulting function.

Example

Find the area contained between $y = -e^x$ and $y = \sin x$ from $x = -\pi$ to $x = \pi$.

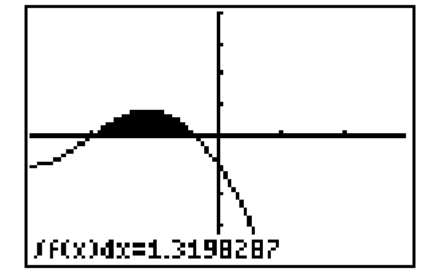


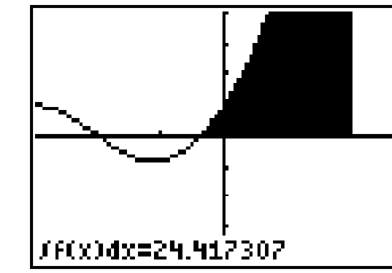
Note that these graphs cross within the given interval. So, we need to find the intersection and then treat each part separately (as the curve that is the upper one changes within the interval).



So we need to find

$$\int_{-\pi}^{-0.589} -e^x - \sin x \, dx \quad \text{and} \quad \int_{-0.589}^{\pi} \sin x - (-e^x) \, dx.$$





The total area $= 1.32\ldots + 24.4\ldots = 25.7$ (3 sf)

Area between a curve and the *y*-axis

Mostly we are concerned with the area bounded by a curve and the *x*-axis. However, for some functions it is more relevant to consider the area between the curve and the *y*-axis. This is particularly pertinent when volumes of revolution are considered in Chapter 16.

The area between a curve and the *x*-axis is $\int_a^b y \, dx$

To find the area between a curve and the *y*-axis we calculate

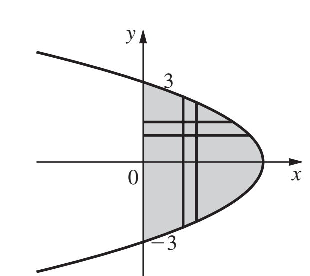
$$\int_a^b x \, dy$$

This is where *x* is a function of *y*.

This formula is proved in an identical way to the area between the curve and the *x*-axis, except that thin horizontal rectangles of length *x* and thickness δy are used.

Example

Consider $y^2 = 9 - x$.



There is no choice here but to use horizontal strips as opposed to vertical strips, as vertical strips would have the curve at both ends, and hence the length of the rectangle would no longer be y and the formula would no longer work.

Area

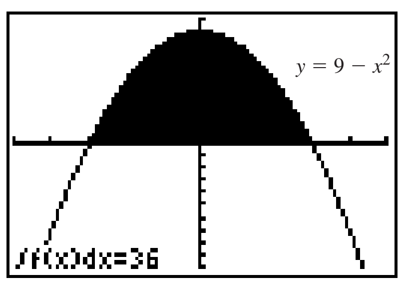
$$= \int_{-3}^{3} x \, dy$$

$$= \int_{-3}^{3} 9 - y^2 \, dy$$

$$= \left[9y - \frac{1}{3}y^3\right]_{-3}^{3}$$

$= (27 - 9) - (-27 + 9)$
$= 18 + 18$
$= 36$ square units

Although the integration is performed with respect to y, a calculator can still be used to find the area (although of course it is not the correct graph).

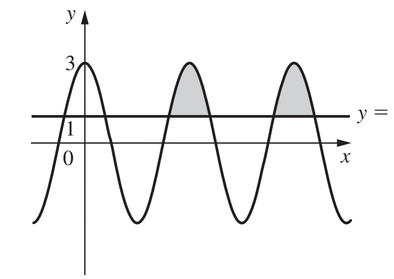


Exercise 9

Calculate the area enclosed by the two functions.

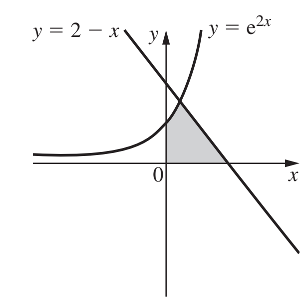
1 $y = x^2$, $y = x$
2 $y = 6x^2$, $y = 3x$
3 $y = x^3$, $y = x$
4 $y = x^2$, $y = \sqrt{x}$
5 $y = 8 - x^2$, $y = 2 - x$
6 $y = x^3 + 24$, $y = 3x^2 + 10x$
7 $y = 10 - x^2$, $y = 19 - 2x^2$
8 $y = e^x$, $y = 4 - x^2$
9 $y = -\frac{1}{2}x^2 + 6x - 10$, $y = 4x - \frac{1}{3}x^2$ and the x-axis. In this case draw the graphs and shade the area.

10 $y = 3\cos 2x$ and $y = 1$ produce an infinite pattern as shown.

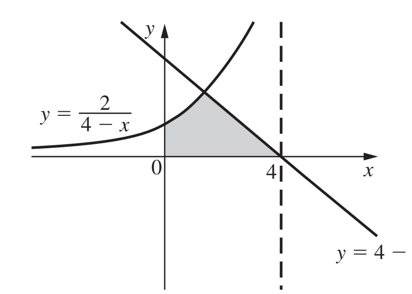


Find the area of each shaded part.

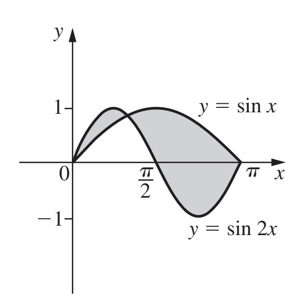
11 Find the area between $y = e^{2x}$, $y = 2 - x$, the x-axis and the y-axis as shown.



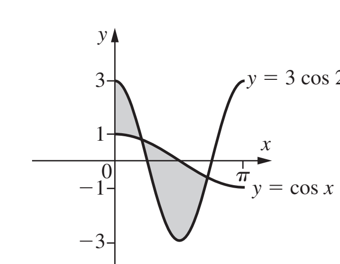
12 Find the area between $y = \frac{2}{4 - x}$, $y = 4 - x$, the x-axis and the y-axis as shown.



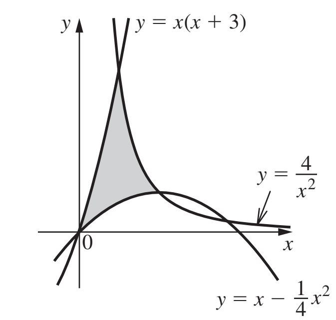
13 Find the shaded area.



14 Find the shaded area.

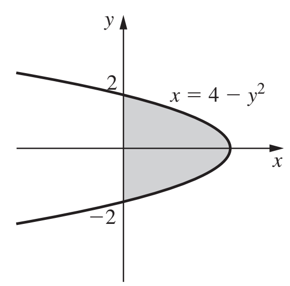


15 Find the area of the "curved triangle" shown below, the sides of which lie on the curves with equations $y = x(x + 3)$, $y = x - \frac{1}{4}x^2$ and $y = \frac{4}{x^2}$.



Find the area enclosed by the y-axis and the following curves.

16 $x = 4 - y^2$

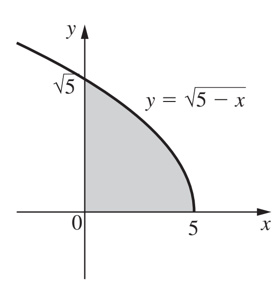


17 $y^2 = 16 - x$

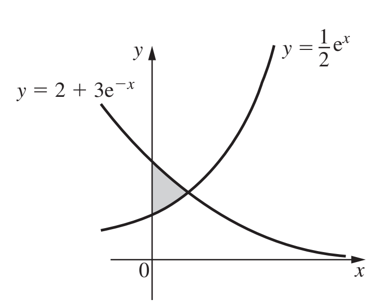
18 $8y^2 = 18 - 2x$

19 Evaluate the shaded area **i** with respect to x and **ii** with respect to y.

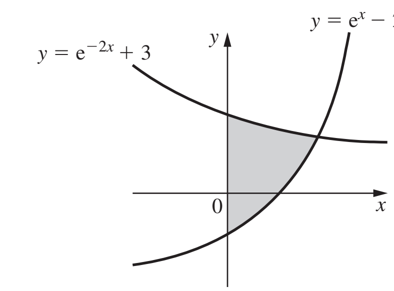
$y = \sqrt{5 - x}$



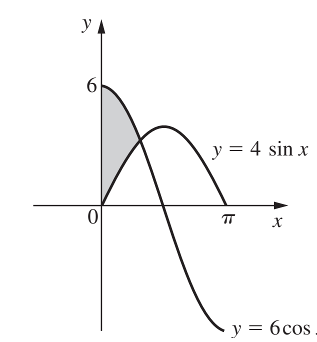
20 Find the shaded area.



21 Find the shaded area.



22 Find the shaded area.



Review exercise

1 Integrate these functions.

a $4x^2 - 7$ **b** $9x^2 + 4x - 5$ **c** $\frac{8}{x^3}$ **d** $(3 - 2x)^2$

2 Solve these equations.

a $\frac{dy}{dx} = \frac{x^3 - 6}{x^5}$ **b** $\frac{dy}{dp} = p^2(3 - p^5)$ **c** $\frac{dy}{dt} = \frac{3t^2 - \sqrt{t}}{4t}$

3 Given $\frac{dy}{dx} = -3x + 8$ and the curve passes through (2, 8), find the equation of the curve.

4 Find these integrals.

a $\int 4e^x - \sin x \, dx$ **b** $\int 7\cos x - \frac{4}{x} dx$ **c** $\int 2e^{6x} - \frac{5}{x} + 4\sin x \, dx$

5 Find these integrals.

a $\int 6\cos 2x \, dx$ **b** $\int 4e^{2x} \, dx$ **c** $\int \frac{2}{4x - 3} dx$

d $\int (3x - 2)^6 \, dx$ **e** $\int 7e^{3x} - \frac{4}{(3x - 4)^5} dx$

6 Let $f(x) = \sqrt{x}\left(2x - \frac{3}{x^{\frac{5}{2}}}\right)$. Find $\int f(x) \, dx$.

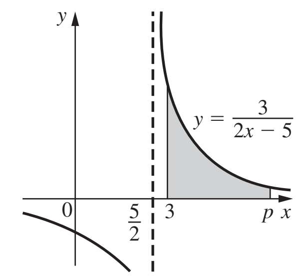
7 Find these definite integrals.

a $\int_1^3 4p - \frac{3}{(2p - 1)^3} dp$ **b** $\int_0^{\frac{\pi}{4}} \sin 4\theta + 1 \, d\theta$ **c** $\int_{-k}^{k} 3\cos 2\theta - 4\theta \, d\theta$

8 Find the area given by these definite integrals.

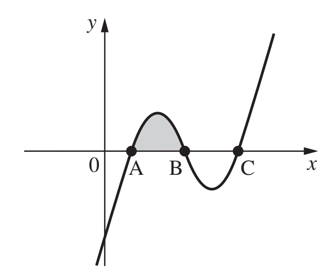
a $\int_2^5 e^x \, dx$ **b** $\int_{-3}^1 4x - 5 \, dx$ **c** $\int_0^{\frac{\pi}{6}} 2 \cos 3\theta \, d\theta$

9 Find an expression in terms of p for this shaded area.



10 Find the total area of the two regions enclosed by the curve $y = x^3 - 3x^2 - 9x + 27$ and the line $y = x + 3$. [IB Nov 04 P1 Q14]

11 The figure below shows part of the curve $y = x^3 - 7x^2 + 14x - 7$.
The curve crosses the x-axis at the points A, B and C.



a Find the x-coordinate of A.
b Find the x-coordinate of B.
c Find the area of the shaded region. [IB May 02 P1 Q13]

12 Find the area bounded by the curve and the x-axis for:

a $y = 3 \cos 2\theta$, $\theta = 0$ and $\theta = \dfrac{3\pi}{4}$

b $y = x^4 - 2e^x$, $x = -1$ and $x = 3$

13 Find the area enclosed by:

a $y = e^x$ and $y = 6 - x^2$

b $y = \dfrac{3}{2 - x}$, $y = 2 - x$ and the x- and y-axes

14 Find the area between $x = 8 - y^2$ and the y-axis.